Characterization Of Periodic Orbits In Open Nonlinear Dynamical Systems

A THESIS SUBMITTED FOR THE DEGREE OF
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by
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Dedicated to

My Parents

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PUBLICATIONS

PAPERS TO BE INCLUDED IN THE PH.D. THESIS

- **Isochronicity and limit cycle oscillation in chemical systems**, Sandip Saha and Gautam Gangopadhyay, J. Math. Chem., Springer, 55 (3), 887–910 (2017).

- **When an oscillating center in an open system undergoes power law decay**, Sandip Saha and Gautam Gangopadhyay, J. Math. Chem., Springer, 57 (3), 750–768 (2018).

- **Reduction of kinetic equations to Liénard-Levinson-Smith Form: Counting Limit Cycles**, Sandip Saha, Gautam Gangopadhyay and Deb Shankar Ray, Int. J. Appl. Comput. Math., Springer, 5 (2) (2019).

- **Systematic designing of bi-rhythmic and tri-rhythmic models in families of Van der Pol and Rayleigh oscillators**, Sandip Saha, Gautam Gangopadhyay and Deb Shankar Ray, Communications in Nonlinear Science and Numerical Simulation, Elsevier, 85, 105234 (2020).

- **Suppressing bi-rhythmicity by parametrically modulating nonlinearity in limit cycle systems**, Sandip Saha, Sagar Chakraborty and Gautam Gangopadhyay *(Submitted)* [arXiv:2007.14883].

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INTRODUCTION

1.1 INTRODUCTION

Poincaré had discovered limit cycle as a close trajectory which causes the system to return to it in the context of his classic exploration on "Celestial mechanics" of three-body problems[1, 2], some of which was formally analyzed by Birkhoff in his famous book[3] on "Dynamical system" and the legacy continued with several other workers[4–6] to trace the origin of auto-oscillations of open natural systems. Self-oscillation as a Rayleigh oscillator[7–9] or Poincaré limit cycle in open dynamical system[10–13] in contrast to ordinary oscillation does not need a periodic driving force from outside. The nature or form of self-oscillation depends on many features of the system but not on the initial condition. Baron Rayleigh[8, 9] due to his keen interest for acoustic instruments, vibration of strings, motion of air in organ pipes and electrical sound generation systems, propounded the basic oscillatory nature of such systems. A remarkable interest in the aesthetics of human perception coupled with his mathematical talent ultimately he arrived at the theory of self-oscillation as the combined effect of nonlinear processes in dissipation and maintenance of vibrational energy through proper shape and size of the instrument[8, 9]. As a young subject it was perpetuated by many others[10–13] to understand the mechanism as a part of feedback loop starting from the work of Van der Pol in radio physics and engineering which brought the revolution in the theory of nonlinear oscillations[10, 14]. Basic model in this scientific area of further exploration is precisely the Rayleigh equation and remarkably it is still standing up with its enigma of nonlinear nature of dissipation which originally needed an extraordinary perception in grasping the physical nature of the problem[15–17]. The undamped oscillations have also been expressed in chemical and biological systems, for example, Belousov–Zhabotinsky (BZ) reaction[15, 18–23], periodic processes in photosynthesis[24], cell biology[25, 26], Glycolysis[6, 27, 28], Calcium oscillation[29] and neural activities[30–32]. The theory of chemical oscillation by Prigogine and Lefever, a Brusselator model[5, 33] as a tri-molecular chemical mechanism which establishes a compatibility of the nonlinear behaviour based on the chemical kinetic process. So the Brusselator model of chemical oscillation, self-oscillation of Rayleigh equation in acoustics and Van der Pol oscillator of radio engineering crosses all their limits of their field of applicability and emerged as generic models of nonlinear phenomena[34–36] in natural processes. Self-oscillation of substrates and products of Glycolytic reaction is one of the most important metabolic pathway of production of energy of living world[29, 37]. Other periodic orbits from localised in space to spatiotemporal domain, namely, traveling waves[36, 38] in biophysical systems[29], all have been paving the path into physical and mathematical biology. So it is quite expected that people have already found the common base of self-sustained oscillation of Rayleigh oscillator of sound and chemical oscillator of Brusselator and Sel'kov models[6] and they are shown to be reduced to the generalized Rayleigh equation[16]. More generally, Liénard[10, 39–43] equation for limit cycle which also covers generalized Rayleigh equation can also be utilzed to include the Van der Pol type equations to have the Liénard–Levinson–Smith (LLS) equation[41–44] or sometimes called the generalised Liénard equation and they are more suitable for arbitrary chemical and biophysical feedback systems.

A periodic orbit in an open nonlinear dynamical system described by two-dimensional ordinary differential equation (ODE) stemming from various phenomenological sources is one of the most important motivations to study nonlinear dynamics. As ordinary perturbation theory often fails due to non-convergence of the series so in order to extract information through perturbation theory there is a need to develop proper techniques to tackle the summation of otherwise divergent series. In this circumstance one has to look for various singular perturbation techniques. All these methods basically demands that at every order of perturbation the so called secular or divergent terms arising out of straightforward application of perturbation theory be removed. It has proved to be a successful tool in finding approximate solutions to weakly nonlinear differential equations with finite oscillatory period. In the multi-scale perturbative treatment[45–49] of dynamical systems the amplitude and phase of the oscillation get renormalized[46–48] and the perturbative series is uniformly valid and does not have any secular term. On the other hand, self-sustained chemical oscillations[33, 36, 50] are also regularly observed in biological world to maintain a cyclic steady state e.g., cell division[12, 28], Circadian oscillation[51], Calcium oscillations[52] and other bio-systems[11, 12, 36]. The generic feature of such diverse nature of nonlinear oscillations are due to autocatalysis and various feedback mechanisms into the system which are basically controlled by a few slow time scales of the overall process. The examples of cyclic dynamics came to the purview of natural phenomena from living systems to earth, atmosphere and heavenly bodies. Such periodic orbits can be isochronous or the frequency may depend on their amplitude of which the most common examples of periodic orbits of open systems are limit cycle and in some special cases they become center. The ubiquity of limit cycle[10] in dynamical system[10, 39, 53] described by a pair of ODEs are quite characterized mathematically, however, a general prescription of shape, size and the number of stable limit cycles in a given system are not yet well established. From the physical point of view the response properties of a limit cycle due to parametrically excited by an external field is also ill understood unlike ordinary external driving in various physical processes. On the one hand there is a challenge in dealing with limit cycles in a strongly nonlinear systems inspite of several developments of various multi-scale perturbation techniques like, Krylov–Bogolyubov (K-B) method[10, 34, 44, 54], Lindstedt-Poincaré method[34, 53], Renormalisation Group (RG) method[46, 48, 49] etc. A limit cycle on the other hand in a given dynamical system can be a great tool as the nature is playing through various stable limit cycles to regulate its self-organized processes which need to be understood, for example, the three synchronized cyclic orbits called the baroreceptor loop is a global feedback control mechanism of heart, lungs and the nervous system in the brain to adjust the heart rate, the venous pressure in order to maintain the arterial pressure at a given level for the ultimate goal of regulating the cardiac output[13]. As a common underlying thread in the thesis we explore a class of open natural dynamical systems which is utilized for the characterization of various periodic orbits. In order to understand the response properties of a limit cycle under external parametric perturbation we have investigated the effect of delay and subharmonic resonances. As the multiple limit cycles in a given system is an important knotty issue, we have investigated on the counting of limit cycles and its application in systematic construction of multi-rhythmic oscillators[39, 55, 56] from a simple limit cycle system. In the present thesis entitled, "Characterization Of Periodic Orbits In Open Nonlinear Dynamical Systems" we have focussed on the following topics:

Isochronicity and limit cycle oscillation in chemical systems: In order to bypass the exact solution of the nonlinear dynamical systems the general trend is to resort to a geometrical approach coupled with tools of analysis[3, 57–62]. To deal with the periodic orbits in open nonlinear dynamical systems one needs to understand the suitability of the various singular perturbation techniques[3, 57, 63]. They have been proved to be successful tools in finding approximate solutions to weakly nonlinear differential equations with finite oscillatory period. In the perturbative renormalization treatment of dynamical systems the amplitude and phase of the oscillation get renormalized[35, 45]. When the approximate solution is expressed in terms of these renormalized amplitude and phase then the perturbative series is uniformly valid and does not have any secular term[46, 47]. In the traditional RG approach [45] in field theory the order parameters designated by various coupling constants play similar role as amplitude and phase in the case of oscillatory dynamical system. RG method also deals with the problem of isochronous centers characteristic of a family of initial condition dependent periodic orbits[34, 48, 49, 58, 64] surrounding a critical point. Isochronicity is a widely studied subject not only for its relation with stability theory and bifurcation theory[11, 12, 63, 65] but also in the study of bifurcations of critical points leading to limit cycles[40] and isochronous systems[48, 58, 64].

Various kinds of periodic trajectories in phase space can be found and most striking example is a limit cycle in the system. A center refers to a family of initial-condition-dependent periodic orbits surrounding a point. While centers can exist in both linear and nonlinear systems, limit cycles can occur only in nonlinear systems. The most important kind of limit cycle is the stable limit cycle where all nearby trajectories spiral towards the isolated orbit. Existence of a stable limit cycle in a dynamical system means there exists self-sustained oscillation in the system. Dynamical systems capable of having limit cycle oscillations are very important from the point of view of modelling real-world systems which exhibit self-sustained oscillations. Some examples of such phenomena from nature include heart beating[66, 67], oscillations in body temperature[68, 69], random eye movement oscillations during sleep[70], hormone secretion[71, 72], chemical reactions that oscillate spontaneously[20, 33, 73, 74] etc. Chemical oscillation[36, 50] is an interesting nonlinear dynamical phenomenon which arises due to intrinsic instability of the non-equilibrium steady state of a reaction under far away from equilibrium condition[33]. Experimentally such open systems like, Bray[75, 76], BZ[15, 18–22] and Glycolytic reactions[6, 27, 72, 77–79] are studied extensively in a continuously flowing stirred tank reactor and the nature of the oscillatory kinetics of two intermediates gave reliable dynamical models of limit cycle. Although a lot of work has been performed to explore the ways to determine if a system has a stable limit cycle, surprisingly a little is known about how to locate this and still it remains a highly active area of research[10, 28, 36, 80]. By casting a class of chemical oscillations usually governed by two-variable kinetic equations into the form of a Liénard oscillator here we have found the conditions of limit cycle and isochronicity in a unified way.

When an oscillating center in an open system undergoes power law decay: Dynamical systems[3, 10, 34, 54, 57, 61, 62] capable of having isochronous oscillations[48, 64, 80, 81] are very important from the point of view of modelling real-world systems[40, 48, 64] which exhibit self-sustained oscillation[11, 12, 63, 65]. The chemical oscillations[33, 36, 50, 74] are also of immense importance in biological world to maintain a cyclic steady state e.g., Glycolytic oscillations[28, 77–79], Calcium oscillations[52], cell division[82], Circadian oscillation[51, 83] and others[36]. To obtain the nonlinear dynamical features of a periodic orbit the general trend is to resort to a geometrical approach coupled with tools of analysis[3, 34, 54, 57, 61, 62]. Recently RG analysis[46, 47] is heavily used to probe the multi-scale oscillation in the nonlinear system. Here a class of arbitrary autonomous kinetic equations in two variables are cast into the form of a Liénard–Levinson–Smith (LLS) oscillator[34, 40, 48, 49, 64] characterized by the nonlinear forcing and damping coefficients which can provide a unified approach to many problems concerning the existence of limit cycle and center. By suitably adopting K-B averaging method in multi-scale perturbation theory for a periodic system here we have explored the solution of a class of two variable open systems through LLS form.

Reduction of kinetic equations to Liénard–Levinson–Smith (LLS) form: counting limit cycles: Various open kinetic systems[10–12, 28, 36, 44] in physics, chemistry and biology, are generically described by a minimal model of autonomous coupled differential equations[3, 34, 54, 59, 60, 84] of two variables. They exhibit self-sustained oscillation in the form of stable limit cycle in a phase plane in many examples, such as, chemical reactions[15, 36, 40], biological rhythms[10–12, 55, 56, 85], vibrations in mechanical[86], optical system and musical instrument[10, 17], to name a few. A Rayleigh[17] equation in violin string and Van der Pol oscillation in electric circuit are the classic examples, in this context. More generally, Liénard[10, 39–43] equation underlines the concrete criteria for the existence of at least one limit cycle for a general class of which covers Van der Pol and Liénard equation is the LLS equation[41–44], sometimes called the generalised Liénard equation. Casting a general system of kinetic equations in two variables which describe a variety of scenarios in physical, chemical, biochemical and ecological sciences into LLS form[44] is often not straight-forward[10, 40, 44, 87]. In this context we have presented a scheme to cast a set of a class of coupled nonlinear equations in two variables into a LLS form so that the later becomes applicable to nonlinear dynamics of open feedback systems including the the counting problem of number of limit cycles.

Systematic designing of bi-rhythmic and tri-rhythmic models in families of Van der Pol and Rayleigh oscillators: Dissipative nonlinear dynamical systems[10, 12, 28, 34, 36, 39, 44, 54] are often described by minimal models governed by autonomous coupled differential equations[34, 54] which admit of periodic orbits in the form of limit cycles in two-dimensional phase space[10, 34, 39]. This kind of self-excited periodic motion is generically distinct from the forced or parametric oscillation[17] and arises as an instability of motion when the dynamical system at a steady state is subjected to a small perturbation. The self-excited oscillation is well known in heart beat[66, 88], nerve impulse propagation through neurons[31, 32], Circadian oscillation[51] or sleep-wake cycle, Glycolytic oscillation[6, 78, 79, 87] in controlling metabolic activity in a living cell and many other biological phenomena[12, 28, 55, 56, 85, 89–92]. Several musical instruments and human voice in acoustics, lasers in radiation-matter interaction, electrical circuits involving nonlinear vacuum tubes, oscillatory chemical reactions[15, 36, 40] concern self-excited oscillations. Apart from understanding these various phenomena in terms of the limit cycle solutions of the nonlinear differential equations, self-excited oscillations are also interesting from the point of view of energetics, as they are not induced by any external periodic forcing[17, 93]; rather the oscillation itself controls the driving force to act in phase with velocity. It results a negative damping stuation which acts as an energy source of oscillation. From the perspective of energetics or thermodynamics, as emphasized in Ref. [88], it is useful to regard the mathematical limit cycle as representing a "thermodynamic cycle" in which the thermodynamic state of the system varies in time but repeats over a cycle after a finite period. This requires an active non-conservative force (effectively the anti-damping) as well as a load (the nonlinear damping) in the dynamical system. This aspect of self-oscillation has been successfully utilized to revisit the concept of thermodynamic heat engine in a solar cell[94] and in electron shuttle[95], a model nano-scale system. It has been shown [16, 40, 87, 96] that for a wide class of kinetic models describing biological and chemical oscillations in two-dimensional phase space variables can be cast in the form of either Rayleigh or Van der Pol oscillator or any of their generalizations[16, 40–44, 87, 96, 97]. Both the oscillators, however, can be subsumed into a common form, i.e., LLS oscillator[16, 40–44, 87, 96, 97], so that they can be viewed as the two special cases[96] of LLS system. While the standard Rayleigh or Van der Pol oscillator allows single limit cycle, because of polynomial nature of nonlinear damping force and restoring force functions, LLS system exhibits multi-rhythmicity[98, 99], i.e., one observes the co-existence of multiple limit cycles in the dynamical system. In some biological systems nature utilizes this multi-rhythmicity as models of regulation and in various auto-organisation of cell signalling[28, 98–101]. In a related issue a bi-rhythmic model for Glycolytic oscillation was proposed by Decorly and Goldbeter[102]. The coupling of two cellular oscillations[99] also leads to multi-rhythmicity. By extending Van der Pol oscillator Kaiser had suggested a bi-rhythmic model[55, 85] which has subsequently been used in several occasions[56, 89–92, 103]. As there is no general way to obtain such multi-rhythmicity, here based on a general scheme of counting limit cycles of a given LLS equation we have proposed a recipe for systematically designing models of multi-rhythmicity.

On the suppression of bi-rhythmicity by parametrically modulating nonlinearity in limit cycle systems: Since Faraday's observation [104] of parametric oscillations as surface waves in a wine glass tapped rhythmically, almost two centuries have passed and over the years, it has been realized that the phenomenon of parametric oscillations is literally omnipresent [53, 105] in physical, chemical, biological, and engineering systems. Parametric oscillations are essentially effected by periodically varying a parameter of an oscillator which, thus, is aptly called a parametric oscillator. The simplest textbook example with wide range of practical applications is the Mathieu oscillator [106] where the natural frequency of a simple harmonic oscillator is varied sinusoidally and the interesting phenomenon of parametric resonance [107] is observed. The effect of additional nonlinearity in the Mathieu oscillator has also been extensively investigated, e.g, in Mathieu–Duffing [108] and Mathieu–Van der Pol–Duffing [109] oscillators. However, only rather recently, the effect of periodically modulating the nonlinearity in a limit cycle system, viz., Van der Pol oscillator has been investigated [93]. The resulting parametric oscillator, termed PENVO (**p**arametrically **e**xcited **n**onlinearity in the **V**an der Pol **o**scillator), along with the standard phenomenon of resonance, exhibits the phenomenon of antiresonance that is said to have occurred if there is a decrease in the amplitude of the limit cycle at a certain frequency of the parametrical drive.

In the context of the limit cycle oscillations [88], one often comes accross the systems possessing more than one stable limit cycle. Such multicycle systems are manifested in biochemical processes [6, 78, 110–115]; one of the simplest of them being a multicycle version of the Van der Pol oscillator [55, 85, 89] modelling some biochemical enzymatic reactions. This oscillator has two stable limit cycles (and an unstable limit cycle between them in the corresponding two-dimensional phase space) owing to the state dependent damping coefficient that has up to sextic order terms. Consequently, it shows bi-rhythmic behaviour wherein depending on the initial conditions, the long term asymptotic solution of the oscillator corresponds to one of the stable limit cycles that have, in general, different frequencies and amplitudes. Needless to say, bi-rhythmicity is a widely found phenomenon across disciplines—and not only in biochemical processes—because so are the ubiquitous limit cycle oscillations. How to control bi-rhythmicity in a nonlinear system is interesting as a nonlinear phenomenon. Here we have illustrated that the bi-rhythmicity seen in the delayed Van der Pol oscillator and its modified version, the Kaiser oscillator to include higher order nonlinear damping, can be suppressed if the nonlinear terms of the oscillators are periodically modulated.

1.2 SCOPE OF THE THESIS

Although the scope of the work on the characterization of periodic orbits in open nonlinear dynamical system is a very broad, here we shall confine ourselves within the limit of a few aspects of such types of motion. Dynamical systems are usually described by ODEs or partial differential equations but more general dynamical systems, such as, maps, delay equations, stochastic differential equations, etc. are also considered. A nonlinear system is a system that does not satisfy the superposition principle. Most importantly, nonlinear systems may contain multiple attractors, each with its own basin of attraction. Thus the fate of a nonlinear dynamical system may depend on its initial state and a whole new set of phenomena arises associated with the way in which basins of attraction shift as parameters are varied. Nonlinearity can also give rise to an entirely new types of attractors, for example, Lorenz and Rössler attractors etc[10]. Limit cycles in nonlinear systems may be quite complicated, circling around in a bounded region of state space many times before finally closing on themselves. It is even possible and quite common for a trajectory to be confined to a region of state space where there are no stable limit cycle or fixed point. For nonlinear systems multiple solutions can exist which are visited through bifurcations when a parameter of the system is varied and can lead to interesting phenomena such as chaos. Dynamical systems can model an incredible range of behaviour such as the motion of planets in the solar systems, the way diseases spread in a population, the shape and growth of plants, the interaction of optical pulses, or the processes that regulate electronic circuits and heart beats.

To put our work in the proper perspective here we have made a brief survey of the literature of the dynamical systems, specially on the limit cycle and other periodic orbits of nonlinear systems and their analysis through multi-scale perturbation theories along with probing them with the parametric excitation.

French mathematician Henri Poincaré is the founder of the modern, qualitative theory of dynamical systems. Poincaré published two classical monographs,"New Methods of Celestial Mechanics" (1892–1899)[1] and "Lectures on Celestial Mechanics" (1905–1910)[2] where he successfully applied the results of their research to the problem of the motion of three bodies and studied in detail the behaviour of solutions (frequency, stability, asymptotic, and so on). In 1913, G. D. Birkhoff proved Poincaré's "Last Geometric Theorem", a special case of the three body problem, followed by his book named "Dynamical Systems"[3]. The name of the subject "Dynamical Systems", came from the title of that classic book. Poincaré recurrence theorem states that certain systems will, after a sufficiently long but finite time, return to a state very close to the initial state. He recognized that even differential equations can be viewed as a discrete time systems by strobing, i.e. only recording the solution at a set of discrete times, or by Poincaré section. Aleksandr Lyapunov developed many important approximation methods among which he developed the theory of dynamical stability in 1899 which make it possible to define the stability of sets of ODEs. Smale[60] made significant advances as well. His contribution in the Smale horseshoe has jump-started significant research in dynamical systems. On the periods of discrete dynamical systems Sharkovsky[116] in 1964 implies that if a discrete dynamical system on the real line has a periodic point of period 3, then it must have periodic points of every other period. In the late 20^{th} century, Palestinian mechanical engineer Nayfeh applied nonlinear dynamics in mechanical and engineering systems[53] pioneered the work in applied nonlinear dynamics, in the construction and maintenance of machines and structures that are common in daily life, such as ships, cranes, bridges, buildings, skyscrapers, jet engines, rocket engines, aircraft and spacecraft.

Bifurcation theory considers a structure in phase space (typically a fixed point, a periodic orbit, or an invariant torus) and studies its behaviour as a function of the bifurcation parameter. At the bifurcation point the structure may change its stability, split into new structures, or merge with other structures. By using Taylor series approximations of the maps and an understanding of the differences that may be eliminated by a change of coordinates, it is possible to catalogue the bifurcations of dynamical systems. The Hopf bifurcation theorem has great applications to nonlinear oscillations in circuits and systems[117], for example, nonlinear oscillations of a cantilever tube carrying an incompressible fluid by Bajaj et al.[118], bifurcation appearing in fluid mechanics by Kloeden et al.[119], periodic solutions of second order autonomous ODE by Sabatini[120] and Swift[121] with the symmetry of the square. Li et al.[122] have studied Zero-Hopf bifurcation and Hopf bifurcation and the distribution of the equilibrium points of the modified Chua system[123].

Limit cycling phenomena[124] can be observed in many electrochemical and mechanical systems, in predator-prey communities[125], in controlling the multiplicity of limit cycles near Hopf bifurcation[126], uniqueness and non-existence of limit cycles[127, 128]. Bifurcation analysis are studied with harmonic balance[129], co-dim 1 bifurcation[130] along with the the stability of limit cycles in hybrid systems[131]. Gérard and Goldbeter[132] have shown that the cell cycle (mammalian) behaves as a limit cycle oscillator. Limit cycle in presence of noise are also studied in Ref. [133], the entrainment of noise induced limit cycle oscillator[134] both for additive and multiplicative noise and robustness of periodic orbits in the presence of noise[135]. Recently, the limit cycle has been the central issue in precision control designs as the need of precision positioning systems becomes unavoidable[136–138]. Functional analysis[139] is reported in analyzing the limit cycles in nonlinear systems and Floquet theory[140] and contraction analysis is reviewed for synchronization and coordination of coupled oscillators.

Initial works in the area of isochronous oscillators are traced back to Galileo Galilei and Christian Huygens[141]. They investigated the problem of achieving perfect isochronicity and showed that it can be realized in a simple pendulum that wraps around the cycloid[142]. More recent investigations of isochronicity have been directed towards nonlinear oscillators, by Gasull et al.[143] studied the center and isochronicity conditions for systems with homogeneous nonlinearities. Some Liénard-type equations are found to exhibit the isochronicity characteristic is also analyzed in some cases by Sabatini[144]. Necessary and sufficient mathematical conditions are analyzed by Christopher et al.[145], Chandrasekar et al.[146, 147]for the isochronicity of the differential equation. Sabatini[148] studied the analogous equation and later in Refs. [149–152]. Isochronicity of centers are also studied in Refs. [153, 154]. For mechanical oscillators[155] including homogeneous systems, and hamiltonian systems[156–158] isochronous center has been carried out.

A new technique to manufacture isochronous hamiltonian systems has been introduced by Calogero et al.[159–162]. They also introduced a new class of isochronous dynamical systems[163] to describe chemical reactions[164] also the isochronous oscillators as well as isochronous dynamical systems. Methods to extend any dynamical system such that it becomes isochronous or asymptotically isochronous or multi-periodic have been shown by them later[165]. In a recent investigation, Sarker et al.[49] have shown that RG technique can be considered as a probe for center or limit cycle through amplitude and phase equation. It has a non-trivial ability of classifying the solutions into limit cycles and periodic orbits surrounding a center. They also showed that the methodology has a definite advantage over linear stability analysis in analyzing centers as well as isochronous orbits. Later Sarkar et al.[166] studied the condition of isochronicity for two-dimensional systems in the RG context where they found a necessary condition for the isochronicity of the Cherkas system[167] and another class of cubic system.

The uniqueness and existence of periodic solutions of the Liénard equation[40–44] has been analyzed by Staude[168] in a detailed manner. Poincaré–Bendixson domain has been constructed to prove the existence of at least one periodic solution. Liénard condition, a sufficient condition for the existence of a stable limit cycle for a second order equation has been studied by Filippov[169]. The qualitative behaviour of solutions[170] of Liénard equation, Lloyd[171] described in his paper that there is an extensive literature on Liénard's equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ and numerous criteria for the existence of limit cycles have been developed, in the survey of Staude[168], for example. Uniqueness of limit cycles for a class of Liénard systems and with applications has been studied in Ref. [172–175]. Sabatini et al.[176] studied limit cycle's uniqueness for a class of generalized Liénard systems and Hayashi[177] has discussed the non-existence of limit cycles of a Liénard system. An application to the Schnakenberg model[16] of an autocatalytic chemical reaction is given in Ref. [178]. A detailed study of the dynamics of a Liénard System has been done by Velez et al.[179], Rudenok[180] for the generalized symmetry of the Liénard system and by Kukles[181] in the generalized symmetry method.

The first observation of an oscillating chemical reaction in solution phase was made in 1921 by Bray[75], who reported periodic variations in iodine concentrations during decomposition of hydrogen peroxide catalyzed by the iodate ion. The scientific community considered this behaviour as contradicting the second law of thermodynamics and attributed the oscillations to unknown impurities. Prior to about 1920, oscillations in closed homogeneous systems were considered impossible. A crucial discovery was made by the Russian chemist, Belousov[18] in the year 1951, which led the way towards the future development of nonlinear dynamics in chemistry. He observed oscillation of the solution color during the oxidation of citric acid by bromate catalyzed by ceric ions. The final chapter of this story was written by Anatol Zhabotinsky[182] and his first paper on chemical oscillations to the Russian journal Biofizika[183]. After understanding the principal mechanism of this oscillating reaction [184], Zhabotinsky switched his attention to produce chemical waves and it remains a useful tool in many scientific fields, from nonlinear kinetic theory to biological disciplines.

Oscillating chemical reactions and nonlinear dynamics have extensively been studied by R. J. Field, R. M. Noyes, F. W. Schneider, I. R. Epstein and other scientists[20, 185–189]. Despite the theoretical work of Prigogine[5] about oscillations in far from equilibrium systems (1955), the myth that chemical oscillations in homogeneous systems were impossible, because they contradict the second law of thermodynamics, persisted until the mid-1960s (which Zhabotinsky calls the Dark Age). In the year of 1973, Nicolis et al.[190] reported a detailed review containing a synthetic view of the mathematical, thermodynamical, and purely kinetic or experimental work on chemical oscillations[185, 186, 191] which is one of the most spectacular in elementary chemical oscillators [18, 21, 83, 188, 192–195], Continuous Stirred Tank Reactor (CSTR)[36, 50, 196] etc. A general survey of the chemistry and phenomenology of the principal chemical oscillations is followed by a discussion of the situations leading to periodic reactions on the basis of the multi-variable kinetics of feedback systems carried out by Franck[197]. In 1996, Epstein et al.[198] discussed about nonlinear chemical dynamics where chemical reactions with nonlinear kinetic behaviour, including periodic and chaotic changes in concentration, travelling waves of chemical reactivity, and stationary spatial (Turing) patterns.The topics of chemical oscillations, waves, and turbulence is discussed in detail by Kuramoto[74] in his book.

Most nonlinear dynamical systems are handled by various perturbative approaches or asymptotic analysis[3, 57–62]. where the perturbation theory usually confers to a collection of iterative methods for the systematic analysis of global behaviour of differential equations. To deal with non-convergence[3, 57, 63] of the series, multi-scale analysis appears in almost every field of science such as fluid dynamics[199, 200], hydrodynamics [201], mechanics[202], cosmologies with nonlinear structure[203]. Geometric singular perturbation theory for ODEs has been described by Fenichel[204]. Ikegami[205] has discussed geometric singular perturbation theory for electrical circuits. Omohundro[206] investigated the hamiltonian structure of the various perturbation theories used in practice by describing the geometry of a hamiltonian structure for non-singular perturbation theory applied to hamiltonian systems on symplectic manifolds and the connection with singular perturbation techniques based on the method of averaging. Perturbation theory of smooth invariant tori of dynamical systems is discussed by Samoilenko[207]. A singularly perturbed planar system of differential equations modelling an autocatalytic chemical reaction is studied by Gucwa et al.[208] which has a limit cycle where geometric singular perturbation theory is used to prove the existence of this limit cycle. Singular perturbations in noisy dynamical systems has reported by Matkowsky[209]. Perturbation around exact solutions for nonlinear dynamical systems and application to the perturbed Burgers equation is discussed by Irac-Astaud[210]. Conservative perturbation theory for non-conservative systems has been studied by Shah et al.[211] to show how to use canonical perturbation theory for dissipative dynamical systems capable of showing limit cycle oscillations. A detailed survey regarding the perturbation theory in dynamical systems along with various methods have been discussed in Ref. [212, 213].

The phenomenon of parametric instability is frequently encountered in mechanics as well as in various areas of physics. Faraday[104] was one of the first to observe the phenomenon of parametric resonance noting that surface waves in a fluid-filled cylinder under vertical excitation exhibited twice the period of the excitation. Melde [214] was the first to observe it in structural dynamics. Oscillations of a pendulum under parametric excitation was studied by Struble[215]. In 1968, Mathieu[106, 216, 217] observed a phenomenon similar to the one studied by Melde [214] while he was investigating the vibrations of an elliptical membrane by separation of variables when he presented the simplest differential equation that governs the response of many systems to sinusoidal parametric excitation. The chaotic behaviour of a parametrically excited system is studied in some detail by Ariaratnam et al.[218]. Damping of parametrically excited single degree of freedom systems has been studied by Asfar et al.[219]. Blankenship et al.[220] have studied steady state forced response of a mechanical oscillator with combined parametric excitation. Study of parametric excitation in nonlinear dynamical systems using both the harmonic balance method and the normal form method of averaging has been studied by Bakri et al.[221], where they observed Neimark-Sacker bifurcation[10]. Billah[222] provided the definition of parametric excitation for vibration problems where he classified the difference between periodic variation of all parameters of a mechanical system with parametric excitation. Parametric excitation in coupled waves has been studied by Nishikawa[223, 224] by considering a common parametrical coupling between two identical linear oscillators[225, 226]. Also, parametric excitation has been studied in other areas like evolutionary dynamics[227], optomechanical resonators[228], electromechanical systems[229], electrostatically driven micro-electro-mechanical (MEM) oscillators[230], two degree of freedom MEMS system[231], and many others for example in hamiltonian approach by Leroy et al.[232], theory of parametrically excited linear discrete systems[53].

Keeping this background literature in view in the context of Liénard equation which we have specifically used in the study of nonlinear dynamical properties of an open system and it is shown to obtain the condition of limit cycle. In conjunction with the property of limit cycle oscillation, here we have shown the condition for isochronicity for different chemical oscillators with the help of RG method with multiple time scale analysis from a Liénard system. When two variable open system of equations are transformed into a Liénard system we have raised the question about how the condition for limit cycle and isochronicity can be shown in a unified way.

Then, we have probed the condition of periodic oscillation in a class of two variable nonlinear dynamical open systems modeled with LLS equation which can be a limit cycle, center or a very slowly decaying center-type oscillation. Using a variety of examples of open systems like chemical oscillation, population dynamics, optical oscillation in laser and a time delayed nonlinear feedback oscillation as a non-autonomous system, as well as a family of periodic orbits which can be captured in the purview of LLS systems. In terms of a multi-scale perturbation theory, it is utilized to characterize the size and shape of the limit cycle and center as well as the approach to their steady state dynamics.

To cover the case of open chemical oscillations along with Van der pol and Rayleigh systems, we have presented an unified scheme to express a class of system of equations into a LLS equation. We have derived the condition for limit cycle with special reference to Rayleigh and Liénard systems for arbitrary polynomial functions of damping and restoring force. We have shown how a simple multi-scale method can be implemented to determine the maximum number of limit cycles.

Based on such method an algorithm for finding out maximum number of limit cycles possible for a class of LLS oscillator, one can explore a new methodology for computation of system of oscillator with a desired number of limit cycles. For explicit elucidation of real chemical and physical system one needs to develop numerical approach for systematic searching of the parameter space for bi-rhythmic and tri-rhythmic systems and their higher order variants.

One can also characterize limit cycle through the parametric excitation as parametrically excited system is prepared by time-varying coefficients of the equations of motion. Parametric excitation of a system differs from direct forcing as in the former case fluctuations appear as temporal modulation of a parameter rather than as a direct additive term. A common paradigm is that of a pendulum hanging under gravity whose support is subjected to a vertical sinusoidal displacement. In the context of parametric excitation of limit cycle system, there has been no investigation into the control of multi-rhythmicity in a parametric oscillator whose parameter is varied. One should note that periodic variation of such a parameter is understandable [233]. In this context, we used the parametrical excitation phenomenon in some nonlinear time delayed systems where the frequency of parametric excitation is twice that of the natural frequency of the unforced system.

1.3 PLAN OF THE THESIS

Layout of the thesis is as follows:

In chapter 2, we have provided a brief review of the background concepts frequently used in the main text. In the overview we have started from a brief introduction on Rayleigh oscillator and chemical oscillation with their non dimensional mathematical modelling. Then we have provided the basic physical overview of Hopf bifurcation and a short introduction about limit cycle along with a physical example with considering a $\lambda - \omega$ form. A general concept for isochronicity is reviewed. Various perturbative methods such as regular as well as multi-scale perturbative methods are discussed there. The concept of approximate analytical solutions using various methods are shown where in all methods the Van der Pol—Duffing oscillator is considered as a typical example for illustration. The classification of periodic orbits using perturbative methods is also discussed.

In chapter[1] 3, we have briefly reviewed the method of reduction of kinetic equation into Liénard form to find the condition for limit cycle. Isochronicity for various Liénard system is described such as modified Brusselator model, Glycolytic oscillator and Van der Pol type oscillator.

1 Some portion of this chapter is published in the J. Math. Chem.-Saha et al. (2017)

In chapter[2] 4, we have formulated the problem in terms of Liénard–Levinson–Smith (LLS) equation with taking examples of various systems and studied the dynamical consequences of limit cycle, center and slowly decaying center-type oscillations in Glycolytic oscillator, Lotka-Volterra model, Van der Pol type oscillator and a time delayed nonlinear feedback oscillator. There, we have explored the source of power law decay of the center under certain perturbation.

In chapter[3] 5, the problem has been formulated in terms of Liénard–Levinson–Smith (LLS) oscillator to apply perturbation theory where a more compact version is provided to the reduction of kinetic equations to LLS form. Perturbation theory is applied to find maximum number of limit cycles for a LLS system and we have reviewed various model system starting from one cycle cases to multicycle cases.

In chapter[4] 6, the counting of number of limit cycles for polynomial damping and restoring force functions of a Liénard–Levinson–Smith (LLS) system is revisited. The generalisation of single-cycle oscillator to multicycle cases is discussed by reviewing various model systems through the classification of two families — Van der Pol and Rayleigh family of oscillators. Construction of new families of Van der Pol and Rayleigh oscillators with multiple limit cycles (such as bi-rhythmic and tri-rhythmic) and their alternative forms are discussed. Bi-rhythmicity apart from LLS system is also discussed.

In chapter[5] 7, we have discussed the presence of time delayed feedback affects the features of resonance and the antiresonance in the parametrically excited Van der Pol oscillator. Furthermore, we have shown how the resulting bi-rhythmicity therein is suppressed by tuning the strength of the periodic modulation. Subsequently, we consider multicycle Van der Pol oscillator whose nonlinearity parameter is sinusoidally varying and we have shown the way to control bi-rhythmicity in this system as well.

2 Some portion of this chapter is published in the J. Math. Chem.-Saha et al. (2018)
3 Some portion of this chapter is published in the Int. J. Appl. Comp. Math.-Saha et al. (2019)
4 Some portion of this chapter is published in Communications in Nonlinear Science and Numerical Simulation - Saha et al. (2020)
5 Some portion of this chapter is submitted-Saha et al.(submitted) [arXiv:2007.14883]

2

OVERVIEW OF PERIODIC SYSTEMS AND PERTURBATIVE METHODS

2.1 INTRODUCTION

Here we have provided the background concepts and methods on periodic orbits in nonlinear dynamical systems, starting from a brief introduction on Rayleigh oscillator in section 2.2 along with chemical oscillation and their mathematical modelling (in section 2.3). Then we have provided the basic overview of Hopf bifurcation in section 2.4. A short introduction about limit cycle along with a physical example by considering a $\lambda - \omega$ system is given in section 2.5. In section 2.6, a general concept for isochronicity is reviewed. Various perturbative methods such as regular as well as multi-scale perturbative methods are discussed in section 2.7 and section 2.8, respectively. The concept of approximate analytical solutions using various multi-scale perturbative methods are shown in section 2.9. In section 2.10, the Van der Pol—Duffing oscillator in the form of a $\lambda - \omega$ system is shown. Finally, the classification of periodic orbits using perturbative method is discussed in section 2.11.

2.2 RAYLEIGH OSCILLATOR

Since Rayleigh oscillator is an well studied topic about the vibration of string and various musical instruments it has been described in detail[8, 29] in the literature and here we give only a brief representation of the model to elucidate the physical essence and origin of it. Rayleigh[8] examined the problem in the context where a system with friction which is supported by an internal energy source and the force with which this source acts on the oscillating object is assumed to be small. Thus although the dissipative part is nonlinear in the variables, displacement, ξ and velocity, $\dot{\xi}$, restoring force is linear in displacement. Then, the equations of motion generally can be written as

$$m\ddot{\xi}(t) = -a_0\dot{\xi}(t) - a_1\xi(t) + a_2\dot{\xi}(t), \tag{2.1}$$

where a_0 is the coefficient of linear friction, a_1 is the coefficient of elasticity, and a_2 is the coefficient of the velocity-dependent force component. Physically, the last term on the right-hand side of Eq. (2.1) corresponds to the force that is almost suddenly impacted to excite the system by a fast transfer of additional momentum to it. This force is designed by Rayleigh through his innate perception of the workings of the musical instruments which naturally arise in some techniques of eliciting sounds from instruments or in a manner for the cyclical operation of a clock. The solution of linear equation for $a_2 > a_0$ grows exponentially with time, and at a certain time the velocity becomes large, which means, the first term on the right-hand side of Eq. (2.1) is added by a term containing the velocity to the next power, which is directed opposite to the velocity, i.e., proportional to $\dot{\xi}^3$.

Thus the Rayleigh equation is intuitively introduced in the form

$$\ddot{\xi}(t) + \alpha\dot{\xi}(t) + \alpha^{'}\dot{\xi}^3 + \omega^2\xi = 0, \tag{2.2}$$

where the redefined parameters are $\alpha = (a_2 - a_0)/m$, and $\omega^2 = a_1/m$ as well as a new one, $\alpha^{'}$.

For this unusual nature of nonlinear damping term, in Eq. (2.2), Rayleigh showed that in this system stationary oscillations can be found if α and $\alpha^{'}$ have opposite signs. In this case, if α is negative and $\alpha^{'}$ is positive, then that oscillation is stable. Rayleigh also estimated the radius of the self-sustained cirular orbit of stable oscillation for the steady state motion which is presently called a limit cycle. He had also found an approximate solution to Eq. (2.2) with an accuracy to the third order. This basic Rayleigh oscillator is further generalized in several other situations by Van der Pol, Andronov, Liénard, Levinson, Smith and others[10, 15, 34, 41–44, 53, 54].

2.3 CHEMICAL OSCILLATION

A chemical oscillating reaction is a complex mixture of reacting chemical compounds in which concentration of one or more compounds exhibits periodic changes or where sudden changes of properties occur after a predictable induction time[5, 33, 36, 50, 183]. They are a class of reactions which serves as examples of non-equilibrium thermodynamics, resulting the establishment of nonlinear oscillator[34–36].

It is an interesting nonlinear dynamical phenomenon which arises due to intrinsic instability of non equilibrium steady state of reaction under far from equilibrium condition. The self-sustained and resilient chemical oscillation are common in bio-system as they lie at the heart of all biorhythms, as for example, Glycolytic oscillator[6, 27, 28], Brusselator models[5, 33], Circadian oscillation[51, 83] etc. In the earliest 19^{th} century, G. T. Fechner published a report of oscillation in a chemical system about an electrochemical cell that produced an oscillating current in the year of 1828[36, 187].

The BZ reaction is one of several oscillating chemical system whose common element is the inclusion of bromine and malonic acid[15, 18–23]. Over a period of last three decades chemical oscillation has proved itself to be a new area of investigation in nonlinear dynamics. The mechanism for this reaction is very complex and is thought to involve around 18 different steps which have been the subject of a number of research papers[18, 19]. Two key processes both of which are auto-catalytic occur where process A generates bromine, giving the red colour, and process B consumes the bromine to give bromide ions. One of the most common variations on this reaction uses malonic acid $(CH_2(CO_2H)_2)$ as the acid and potassium bromate $(KBrO_3)$ as the source of bromine. The overall equation is:

$$3CH_2(CO_2H)_2 + 4BrO_3^- \rightarrow 4Br^- + 9CO_2 + 6H_2O.$$

Oscillatory chemical reactions[20, 33, 73, 74] are best treated through a dynamical feedback system by a Prigogine and Lefevre[5] called Brusselator which is the model chemical reaction of the transformation of the substrates, A and B into products C and D, namely

$$A + B \rightarrow C + D,$$

as a reaction with steps:

$$\begin{aligned} \mathrm{A} &\xrightarrow{k_1} \mathrm{X} \\ 2\mathrm{X} &\xrightarrow{k_2} 3\mathrm{X} \\ \mathrm{B} + \mathrm{Y} &\xrightarrow{k_3} \mathrm{Y} + \mathrm{C} \\ \mathrm{X} &\xrightarrow{k_4} \mathrm{D}. \end{aligned} \tag{2.3}$$

The most non-trivial step in the Brusselator model (2.3) is the utilization of intermediate substances X and Y satisfying a tri-molecular reaction, which ensures the existence of an oscillatory regime. Considering the substrates to be in excess and the rate constants are equal to unity, the dynamics of the concentrations of the intermediate species is described by equations:

$$\begin{aligned} \frac{dx}{dt} &= a + x^2y - (b+1)x, \\ \frac{dy}{dt} &= bx - x^2y, \end{aligned} \tag{2.4}$$

where x and y are the dimensionless concentrations with a and b are the dimensionless parameters of the reaction.

In the mathematical theory of bifurcations, a Hopf bifurcation also known as a Poincaré–Andronov–Hopf bifurcation[234–236] is one of the most powerful methods for studying periodic solutions as a critical point where a system's stability switches and either a periodic solution arises or a periodic solution looses its stability. More accurately, it is a local bifurcation in which a fixed point of a dynamical system looses stability, as a pair of complex conjugate eigenvalues cross the imaginary axis in the complex plane where the eigenvalues are calculated by linearizing the system around the fixed point. So, for a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary. When the real parts of the eigenvalues are negative the fixed point is called a stable focus. When they cross zero and become positive, the fixed point becomes an unstable focus and then the orbit will be spiralling out. But this change of stability is a local change and the phase portrait sufficiently far from the fixed point will be qualitatively unaffected. Note that the Hopf bifurcation can only appear in such systems of dimension two or higher.

Consider a two-dimensional planar system,

$$\begin{aligned} \dot{x} &= f(x, y; \sigma), \\ \dot{y} &= g(x, y; \sigma), \end{aligned} \tag{2.5}$$

where "σ" is a parameter. Suppose that the considered system has a fixed point, say $(x, y) = (x_0, y_0)$, which may depend on σ. The eigenvalues of the linearized system can be calculated by the Jacobian matrix about the fixed point and suppose that the eigenvalues at the specific fixed point, (x_0, y_0), are $\lambda_{1,2}(\sigma) = \alpha(\sigma) \pm i\beta(\sigma)$. In the theory of dynamical systems, if $\alpha(\sigma_0) < 0$ then the system will be asymptotically stable and unstable if $\alpha(\sigma_0) > 0$. So, if $\alpha(\sigma) = 0$ at a critical value of $\sigma = \sigma_0$ then σ be the bifurcating parameter. To have a Hopf bifurcation, the following conditions have to satisfy are,

1. Non-hyperbolicity condition (conjugate pair of imaginary eigenvalues):
   $\alpha(\sigma_0) = 0,\ \beta(\sigma_0) = \omega \neq 0$, where $sign(\omega) = sign\left[(\partial g/\partial x)|_{\sigma=\sigma_0}(x_0, y_0)\right]$

2. Transversality condition (i.e. the eigenvalues cross the imaginary axis with non-zero speed):
   $\frac{d\alpha(\sigma)}{d\sigma}|_{\sigma=\sigma_0} = \delta \neq 0$

3. Genericity condition:

$$
\begin{aligned}
\Gamma \;=\; & \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) \\
& + \frac{1}{16\omega}\left(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right) \neq 0, \\
& f_{xy} = (\partial^2 f/\partial x \partial y)|_{\sigma=\sigma_0}(x_0, y_0), \ldots
\end{aligned} \tag{2.6}
$$

An unique curve of periodic solutions bifurcates from the origin into the region $\sigma > \sigma_0$ for $\Gamma\delta < 0$ or $\sigma < \sigma_0$ for $\Gamma\delta > 0$. The origin is a stable fixed point for $\sigma - \sigma_0 > 0$ ($\sigma - \sigma_0 < 0$) and an unstable fixed point for $\sigma - \sigma_0 < 0$ ($\sigma - \sigma_0 > 0$) if $\delta < 0$ ($\delta > 0$) whilst the periodic solutions are stable (unstable) if the origin is unstable (stable) on the side of $\sigma = \sigma_0$ where the periodic solutions exist. So, for an unstable fixed point the amplitude of the periodic orbits may exist and grows like $\sqrt{|(\sigma - \sigma_0)|}$ until their periods tend to $2\pi/|\omega|$ as σ tends to σ_0. The bifurcation is called supercritical if the bifurcating periodic solutions are stable, and subcritical if they are unstable.

This two-dimensional version of the Hopf bifurcation theorem was known to Andronov et al.[237] from around 1930, and had been suggested by Poincaré[238] in the early 1890s. Hopf [239], in 1942, proved the result for arbitrary (finite) dimensions. He showed that oscillations near an equilibrium point can be understood by looking at the eigenvalues of the linearized equations for perturbations from equilibrium, and at certain crucial derivatives of the equations. Note that the above conditions for Hopf bifurcation are defined locally around the fixed point (x_0, y_0) which implies that Hopf bifurcation is a local bifurcation. Sometimes the explained conditions failed to provide the system behaviour before or after bifurcation due to higher order nonlinearities and then the concept of central manifold and normal form theory come to fix that kind of specific problems[240]. It also helps us to reduce a higher-dimensional nonlinear systems to a planar one which is essentially provides that apart from the two purely imaginary eigenvalues no other eigenvalues have zero real part[235, 240].

The center manifold theorem is predominantly a model reduction technique for determining the local asymptotic stability of an equilibrium of a dynamical system when its linear part is not hyperbolic. The overall system is asymptotically stable if and only if the center manifold dynamics is asymptotically stable. This allows for a substantial reduction in the dimension of the system whose asymptotic stability must be checked. In fact, the center manifold theorem is used to reduce the system from N-dimensions to 2-dimensions [240]. Moreover, the Center manifold and its dynamics need not be computed exactly; frequently, a low degree approximation is sufficient to determine its stability. Extensions exist to infinite-dimensional problems such as differential delay equations and certain classes of partial differential equations (including the Navier-Stokes equations)[235]. This is some times called Naimark-Sacker bifurcation[235, 241, 242].

2.5 LIMIT CYCLE

Limit cycle is an isolated closed trajectory in phase space exhibited by two-dimensional dynamical systems. As dynamical system evolves, its neighbouring trajectories may either spiral towards or move away from it. If all the neighbouring trajectories move towards the closed trajectory, then the orbit is called a stable limit cycle attractor or periodic attractor and an infinite subset of initial conditions would be drawn to the attractor i.e. a limit cycle attractor is independent of initial condition which constitutes the basin of attraction. On the other hand, an unstable limit cycle (or repeller), is one for which the trajectories are beginning near and will escape away from it towards some neighbouring stable limit cycle attractor or will collapse on some neighbouring stable fixed point. Stable limit cycle implies self-sustained oscillations of a system where the closed trajectory describes perfect periodic behaviour of the system. Any small perturbation from this closed trajectory causes the system to return to it, making the system stick to the limit cycle. The "Poincaré–Bendixson" theorem and "Bendixson–Dulac" theorem predicts the existence of limit cycles for two-dimensional nonlinear systems. The number of limit cycles of a homogeneous polynomial differential equation is the main object of the second part of Hilbert's 16^{th} problem[243].

There are different types of limit cycle oscillations where on the trajectory for which energy of the system would be constant over a cycle i.e. on an average there is no gain or loss of energy. For two-dimensional systems, in general, the limit cycle appears because of the delicate balance of the gain and loss of energy through the nonlinear damping terms corresponding to the nonlinear interaction of the phase space variables. However, some times linear systems under memory effects can provide a similar scenario that produces limit cycles[244].

2.5.1 *Limit cycle as $\lambda - \omega$ system*

Limit cycle as $\lambda - \omega$ system is a class of simple reaction-diffusion equations with a limit cycle in the reaction kinetics system was first introduced by Kopell and Howard[192, 245, 246] to describe a class of examples which appears to exhibit travelling wave phenomena. Later Greenberg[247, 248] showed that asymptotic expansions for spiral waves for $\lambda - \omega$ systems are valid on some specific interval and Ermentrout[249] used it to have stable small-amplitude solutions in reaction-diffusion systems and also Sherratt[250] used to have the evolution of periodic plane waves in reaction-diffusion systems of $\lambda - \omega$ type.

The governing reaction kinetic equations (without diffusion) for $\lambda - \omega$ system are:

$$\begin{aligned} \frac{dx}{dt} &= \lambda(r)x - \omega(r)y, \\ \frac{dy}{dt} &= \omega(r)x + \lambda(r)y; \quad r = \sqrt{x^2 + y^2}. \end{aligned} \tag{2.7}$$

For a coordinate transformation from cartesian to polar with $((x, y) \rightarrow (r, \phi))$ s.t. $(x, y) = (r\cos\phi,\ r\sin\phi)$ with $(r, \phi) = \left(\sqrt{x^2+y^2}, tan^{-1}\frac{y}{x}\right)$ we have

$$\begin{aligned}\frac{dr}{dt} &= r\lambda(r),\\ \frac{d\phi}{dt} &= \omega(r).\end{aligned} \tag{2.8}$$

The non-zero root(s), say, $r = R$, of the amplitude equation confirms the existence of limit cycle(s). The phase solution will be then $\phi = \phi_0 + \omega(R)t$ of frequency $\omega(R)$.

Example 2.5.1. Hopf bifurcation and limit cycle in $\lambda - \omega$ system:

Hopf bifurcation and limit cycle can be described by a pair of equations as $\lambda - \omega$. For illustration, choosing $\lambda(r) = \gamma - r^2$ and $\omega(r) = 1$ in the $\lambda - \omega$ system the reaction dynamics can be written as,

$$\begin{aligned}\frac{dx}{dt} &= (\gamma - (x^2+y^2))x - y,\\ \frac{dy}{dt} &= x + (\gamma - (x^2+y^2))y.\end{aligned} \tag{2.9}$$

It has a stationary point at $(x, y) = (0, 0)$. The form of the above oscillator is very useful to have a stable limit cycle attractor. A wide class of studies have been performed by using the described system in presence or absence of diffusion. If the reaction kinetics is stable limit cycle in nature, then addition of diffusion, provokes the system to provide pattern formation or if the reaction kinetics itself shows up periodic behaviour, then adding diffusion is expected to generate travelling periodic wave-train solution.

So, the characteristic polynomial of the above system at $(0, 0)$ is $1 + (\gamma - K)^2 = 0$ and that implies $K_{1,2} = \gamma \pm i$. Then the fixed point becomes stable for $\gamma < 0$ and unstable for $\gamma > 0$ that means γ is a Hopf bifurcation parameter which bifurcates at $\gamma = \gamma_c = 0$ and the eigenvalues are $\pm i$ satisfying the standard Hopf bifurcation requirement. We have $\omega = 1,\ \delta = 1$ and $\Gamma = -1$, so the bifurcation is supercritical and there is a stable isolated periodic orbit (limit cycle) if $\gamma > 0$ for each sufficiently small γ. Thus for $\gamma = \gamma_c + \epsilon$ with $0 < \epsilon \ll 1$, limit cycle solution with small amplitude can be expected.

2.5.2 *Liénard system*

The Liénard equation is specifically based on the study of nonlinear dynamical systems and ordinary differential equation (ODE), is a special polynomial form of second order ODE, named by the French physicist Alfred–Marie Liénard[41]. It is a initial value problem and it has a lot of applications all over the physical world. During the development of radio and vacuum tube technology, it is intensely studied as it can be used to model oscillating circuits. Liénard described a theorem which was published in 1928 where under certain additional assumptions, Liénard system guarantees the uniqueness and existence of a limit cycle[41]. Let f and g be two continuously differentiable functions, then the second order ODE, $\ddot{x} + f(x)\dot{x} + g(x) = 0$, is called a Liénard equation, where $x(t)$ and the functions f and g satisfying to be an even and odd functions are given below:

1. $x\ g(x) > 0$ for $|x|\ > 0$,
2. $\int_0^{+\infty} g(x)dx = \int_0^{-\infty} g(x)dx = \infty$,
3. There exists some $x_0 > 0$ such that $f(x,v) \geq 0$ for $|x| \geq x_0$,
4. There exists an A such that for $|x| \leq x_0$, $f(x,v) \geq -A$ and
5. There exists some $|x_1| > x_0$ such that $\int_{x_0}^{x_1} f(x,v)dx \geq 10\ A\ x_0$, where $v > 0$ is an arbitrary decreasing positive function of x,

A special form of Liénard equation[41] is, $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, is called the Van der Pol oscillator, is named by an Dutch electrical engineer Balthasar Van der Pol(1889-1959), started his investigation in 1926 and studied version with periodic terms, where chaotic motion can occur[10, 14]. Liénard system also can be solved by Differential Transform method based on the Taylor series expansion by constructing an analytical solution in the form of a polynomial. After that, in the year of 1942, Levinson–Smith modified the Liénard form of equation and wrote the modified version as, $\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0$, known as Levinson–Smith or Liénard–Levinson–Smith (LLS) type oscillatory equation as a generalisation of Liénard oscillator through the damping force function[42, 43]. The property of f and g are the same as described previously. Here we have dealt this type of more general and advanced Liénard equation[41–44]. The Liénard equation where $g(x)$ is a cubic polynomial has been used to describe isotropic turbulence[251]. It is clearly by the pressure of “Nonlinear Damping" coefficient $f(x,\dot{x})$, which depending on the amplitude of oscillation can act as a damping term or pumping term through an internal energy source.

2.6 ISOCHRONICITY

Isochronous systems have been enthralling ever since Galileo’s discovery of the first such system as the simple harmonic oscillator. Famous scientists such as Jacobi, Poincaré, Newton etc. performed experimental investigation of isochronous systems[48, 58, 64]. A sequence of events is isochronous if the events occur regularly or at equal time intervals. The term isochronous is used in several technical contexts, but usually refers to the primary subjects maintaining a constant period. In two-dimensional dynamical systems, isochronicity implies that the frequency is independent of its amplitude, i.e. independent upon the initial conditions. In electrical power generation the word isochronous indicates that the frequency of the electricity generated is constant under varying load. In horology a clock is isochronous if it runs at the same rate regardless of changes in its driving force (also pendulum clocks). In neurology, isochronic tones are regular beats of a single tone used for brainwave entrainment. There is an exhaustive study of various isochronous centers of vector fields in a plane. In the recent years Calogero and Leyvraz[81, 159–165, 252] have opened up new interesting directions in the study of isochronous systems by introducing a simple trick for which one can construct isochronous oscillators by modifying equations for ordinary oscillators.

2.7 PERTURBATIVE METHODS FOR NONLINEAR OSCILLATORS

In this section, we have discussed some basic and necessary perturbative methods for LLS systems that we have used in our work. The basic idea of each perturbative method is to consider a trial solution which may fit in the equation and that provides the approximate solution of a linear as well as a nonlinear system. For a linear system, the solution can be exact, but for a nonlinear system, the solutions comes as an approximate form. The approximations are more accurate if the nonlinearity control parameters are tuned as small as possible. For a nonlinear oscillator, the approximate solution are a combination of lower and higher order harmonics, in a functional form of $\sin(\ldots)$ or $\cos(\ldots)$. The lower order harmonics are called secular terms which is responsible for divergence of the system, which need to be corrected through the amplitude and phase equations. Furthermore higher order harmonics may provide the periodic or aperiodic nature of the system through an approximate solution. Here we have discussed the perturbative methods only for two-dimensional phase space. In a following section we start with a single-scale perturbative method and later some of the multi-scale perturbative methods are also discussed with a common example so that one can easily find the differences.

2.7.1 *Harmonic balance method*

Let us consider a system,

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0, \tag{2.10}$$

with the initial values $(x(0), \dot{x}(0)) = (A, 0)$ and $0 < \epsilon \ll 1$. Now, if there exists a periodic solution (for $\epsilon = 0$) close to $A\cos(\omega t)$ with amplitude A and frequency ω such that the nonlinear function (in presence of ϵ), $h(x, \dot{x}) \approx h\left(A\cos(\omega t), -\omega\, A\sin(\omega t)\right)$ has a Fourier series, then it can be written as,

$$h(x, \dot{x}) \quad = a_1(A)\cos(\omega t) + b_1(A)\sin(\omega t) + \text{higher order terms}, \tag{2.11}$$

where,

$$a_1(A) = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} h(A\cos(\omega t), -\omega\, A\sin(\omega t))\, \cos(\omega t)\, dt \tag{2.12}$$

$$b_1(A) = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} h(A\cos(\omega t), -\omega\, A\, \sin(\omega t))\sin(\omega t)\, dt \tag{2.13}$$

and the constant terms (which are their mean value over a cycle) being zero. Then (2.10) becomes,

$$(1-\omega^2)\, A\cos(\omega t) + \epsilon\, a_1(A)\cos(\omega t) + \epsilon\, b_1(A)\, \sin(\omega t) + \text{higher order terms} = 0. \tag{2.14}$$

So, the equation can hold for all t only if, $(1-\omega^2)A + \epsilon\, a_1(A) = 0$ and $b_1(A) = 0$ and that determines A and ω. If there be a force $\gamma\cos(n\omega t)$ attached with Eq. (2.14) as well as Eq. (2.10) then the periodic response function is equal to $-\frac{\gamma\cos(n\omega t)}{(n^2\omega^2-1)}$, and then the magnitude will be rapidly diminished as large as n increases.

Example 2.7.1. Consider the Van der Pol–Duffing system[109],

$$\ddot{x} + \epsilon(x^2-1)\dot{x} + x - \lambda x^3 = 0; \quad x(0) = A \text{ and } \dot{x}(0) = 0. \tag{2.15}$$

Assuming an approximate solution, $x(t) = A\cos(\omega t)$, and after substituting we have

$$\begin{aligned} &-\frac{1}{4}3A^3\lambda\cos(\omega t) - \frac{1}{4}A^3\lambda\cos(3\omega t) - \frac{1}{4}A^3\omega\epsilon\sin(\omega t) \\ &-\frac{1}{4}A^3\omega\epsilon\sin(3\omega t) - A\omega^2\cos(\omega t) + A\cos(\omega t) + A\omega\epsilon\sin(\omega t) = 0. \end{aligned} \tag{2.16}$$

So, after equating $\cos(\omega t)$ and $\sin(\omega t)$ terms to the both sides we have $-\frac{1}{4}3A^3\lambda - A\omega^2 + A = 0 \implies \omega = \sqrt{1-\frac{3}{4}\lambda A^2}$ and $-\frac{1}{4}A\left(A^2-4\right)\omega\epsilon = 0 \implies A = 2$, where A and ω are the corresponding system amplitude and phase (are in the positive sense), respectively. Finally one can have an approximate periodic solution of the form $A\cos\left(\sqrt{1-\frac{3}{4}\lambda A^2 t}\right)$.

2.7.2 *Lindstedt-Poincaré method of autonomous equations: periodic solutions*

Let us consider a nonlinear ODE,

$$\ddot{x}(t) + \omega^2 \, x(t) = \epsilon \; f(x(t), \dot{x}(t)). \tag{2.17}$$

The system describes an autonomous system which is oscillating with an unknown period T having the nonlinear term, $\epsilon f(x(t), \dot{x}(t))$ that can be treated in perturbation. Upon applying the ordinary perturbation to Eq. (2.17) and writing the solution as a series expansion in ϵ then the secular terms diminish the expansion, and then any predictive power is lost for sufficiently large time.

Redefining a scaled time, $\tau = 2\pi t/T \equiv \Omega \; t$ where T is the (unknown) period to avoid the appearance of the secular terms, then the system (2.17) can be written as,

$$\Omega^2 \frac{d^2 x}{d\tau^2}(\tau) + \omega^2 \; x(\tau) = \epsilon \; f(x(\tau), -\Omega\dot{x}(\tau)) \; . \tag{2.18}$$

One can note that the effect of ϵ both on the solution $x(\tau)$ and the unknown frequency Ω. So, to reduce the effect of the nonlinear term, choosing $0 < \epsilon \ll 1$ and then we can write

$$\begin{aligned} \Omega &= \textstyle\sum_{n=0}^{\infty} \; \epsilon^n \; \alpha_n = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots \quad \text{and} \\ x(\tau) &= \textstyle\sum_{n=0}^{\infty} \; \epsilon^n \; x_n(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \dots \end{aligned}$$

One can expand the nonlinear term, $f(x(\tau), -\Omega\dot{x}(\tau))$ through Taylor series expansion near a critical point or fix it by putting the values of $x(\tau)$ and Ω for a known $f(x(\tau), -\Omega\dot{x}(\tau))$. After expanding Eq. (2.18) by setting all the above considerations one can obtain a system of linear inhomogeneous differential equations where each equation corresponds to a different order of ϵ. The zeroth order equation for ϵ will provide a harmonic oscillator solution. As a result, the first order equation for ϵ will contain the fundamental frequency, corresponding to a period of 2π in the scaled time, and multiples of this obtained frequency will appear through the term f. The presence of a driving term (e.g. parametrical force, external excitation etc.) with the fundamental frequency leads to a resonant behaviour in the first order solution that will contain some regrettable secular terms and then disfigures the expansion. However, it can be dealt with this type of problem by fixing the coefficient α_1 to cancel the resonant term in the r.h.s. of the first order equation of ϵ. Iteration of this procedure to a given order n allows to determine the coefficients $\alpha_0, \dots, \alpha_n$ and therefore the frequency, $\Omega = \alpha_0 + \epsilon \; \alpha_1 + \dots + \epsilon^n \; \alpha_n$ to obtain the approximate analytical solution.

Example 2.7.2. Oscillation of a self-excited system
For example, we choose the Van der Pol–Duffing system (as in previous case),

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x - \lambda x^3 = 0, \tag{2.19}$$

with the initial values $x(0) = a_0$ and $\dot{x}(0) = 0$. For the above system, the frequency, Ω, is unknown, but it may depends on the amplitude and the period will be $\frac{2\pi}{\Omega}$. Let us take a perturbative solution for Ω and $x(t)$ (with a special situation $\lambda = \epsilon$), are,

$$\begin{aligned} \Omega &= 1 + \epsilon\Omega_1 + \dots \\ x(\epsilon, t) &= x_0(t) + \epsilon x_1(t) + \dots \end{aligned} \tag{2.20}$$

The unknown frequency Ω, reduces to 1 when $\lambda = \epsilon = 0$. Replacing, $\Omega t = \tau$, then Eq. (2.19) reduces to

$$\Omega^2\ddot{x} + \Omega\epsilon(x^2 - 1)\dot{x} + x - \epsilon x^3 = 0. \tag{2.21}$$

Now under this substitution, Eq. (2.21) has a known period 2π with

$$x_i(\tau + 2\pi) = x_i(\tau), \ \forall\tau; \ i = 0, 1, \dots\dots \tag{2.22}$$

So, after substituting (2.20), Eq. (2.21) becomes

$$\begin{aligned} &(1 + \epsilon\Omega_1 + \dots)^2(\ddot{x}_0 + \epsilon\ddot{x}_1 + \dots) + \epsilon(1 + \epsilon\Omega_1 + \dots)\left((x_0 + \epsilon x_1 + \dots)^2 - 1\right)(\dot{x}_0 + \epsilon\dot{x}_1 + \dots) \\ &\quad + (x_0 + \epsilon x_1 + \dots) - \epsilon(x_0 + \epsilon x_1 + \dots)^3 = 0 \end{aligned} \tag{2.23}$$

and by assembling powers of ϵ we get

$$\epsilon^0: \ \ddot{x}_0 + x_0 = 0, \tag{2.24}$$

$$\epsilon^1: \ \ddot{x}_1 + x_1 = -2\Omega_1\ddot{x}_0 - x_0^2\dot{x}_0 + \dot{x}_0 + x_0^3, \tag{2.25}$$

and so on. To simplify the calculations we can use the initial values and impose the conditions

$$\begin{aligned} &x_0(0) = a_0, \ \dot{x}_0(0) = 0 \quad \text{and} \\ &x_i(0) = 0, \ \dot{x}_i(0) = 0; i = 1, 2, \dots \end{aligned} \tag{2.26}$$

and then we will get the solution of (2.24) satisfying (2.26) is $x_0(0) = a_0 \cos\tau$. Eq. (2.25) then becomes after putting the value of $x_0(\tau, \epsilon)$

$$\ddot{x}_1 + x_1 = \left(2a_0\Omega_1 + \frac{3a_0^3}{4}\right)\cos\tau + \left(\frac{1}{4}a_0\left(a_0^2 - 4\right)\right)\sin\tau + \frac{a_0^3}{4}\cos(3\tau) + \frac{a_0^3}{4}\sin(3\tau) \tag{2.27}$$

Then, the secular terms i.e. coefficients of $\cos\tau$ and $\sin\tau$ in the first order solution (i.e. ϵ^1) will be periodic only if $\Omega_1 = -\frac{3}{8}a_0^2$ and the amplitude of the period will be $a_0 = 2$. Therefore Eq. (2.27) has a general solution under certain condition is in the form

$$x_1(\tau) = a_1\cos\tau + b_1\sin\tau - \frac{1}{32}a_0^3(\cos(3\tau) + \sin(3\tau)) \tag{2.28}$$

and then by using (2.26) i.e. the initial values, we have,

$$x_1(\tau) = \frac{1}{32}(3a_0^3\sin\tau - a_0^3\sin(3\tau) + a_0^3\cos\tau - a_0^3\cos(3\tau)) \tag{2.29}$$

The final approximate solution of (2.21) will take the form as,

$$x(\epsilon, \tau) \approx a_0\cos(\tau) + \epsilon x_1(\tau) + O(\epsilon^2) \tag{2.30}$$

and after returning to the original time variable t, we have the approximation

$$x(\epsilon, t) \approx a_0\cos(\Omega t) + \epsilon x_1(\Omega t) + O(\epsilon^2) \tag{2.31}$$

where, $\Omega = 1 - \frac{3}{8}\epsilon a_0^2$, gives the dependence of frequency on amplitude.

Example 2.7.3. Forced oscillation of a self-excited system

Let us consider the same Van der Pol–Duffing system with an external forcing term, $F\cos(\Omega t)$, where Ω is the external frequency and F is the magnitude of externally applied force, then we have,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x - \epsilon x^3 = F\cos(\Omega t). \tag{2.32}$$

In the unforced case the system has a limit cycle with an approximate amplitude 2 and period $\frac{2\pi}{\delta}$, $\delta = 1 - \frac{3}{8}\epsilon a_0^2$. For the weak excitation case, F is small and the effect of the external forcing depends on whether Ω is close to the natural frequency or not. If it is close to the natural frequency then an oscillation might be generated which is a perturbation of the limit cycle. But, for the strong forcing case, F would have to be large enough or if the natural and imposed frequency, Ω, are not close enough, and one should expect that the natural oscillation might be quenched like the corresponding linear equation. The analysis only for the case of weak forcing is given below.

Upon rescaling the time, Ωt by τ i.e. $\Omega t = \tau$, one can obtain (from Eq. 2.32),

$$\Omega^2 \ddot{x} + \epsilon\Omega(x^2 - 1)\dot{x} + x - \epsilon x^3 = F \cos\tau. \tag{2.33}$$

As we have considered only the case for weak forcing, one can write $F = \epsilon F_1$. Also taking the perturbative solutions for Ω and $x(t)$ as,

$$\begin{aligned} \Omega &= 1 + \epsilon\Omega_1 + \ldots \\ x(\epsilon, t) &= x_0(t) + \epsilon x_1(t) + \ldots . \end{aligned} \tag{2.34}$$

Then all the above considerations leads to

$$\epsilon^0 : \ \ddot{x}_0 + x_0 = 0 \tag{2.35}$$

$$\epsilon^1 : \ \ddot{x}_1 + x_1 = -2\Omega_1\ddot{x}_0 - x_0^2\dot{x}_0 + \dot{x}_0 + x_0^3 + F_1 \cos\tau \tag{2.36}$$

and so on.
The general solution of (2.35) with period 2π will be

$$x_0 = a_0 \cos\tau + b_0 \sin\tau, \tag{2.37}$$

and after reduction (2.36) becomes,

$$\begin{aligned} \ddot{x}_1 + x_1 = {} & \left(\tfrac{1}{4}r_0^2(3a_0 - b_0) + 2\Omega_1 a_0 + b_0 + F_1\right)\cos\tau + \left(\tfrac{1}{4}r_0^2(3b_0 + a_0) + 2\Omega_1 b_0 - a_0\right)\sin\tau \\ & + \tfrac{1}{4}\left(a_0 + b_0\right)\left(-4a_0b_0 + r_0^2\right)\cos(3\tau) + \tfrac{1}{4}\left(a_0 - b_0\right)\left(4a_0b_0 + r_0^2\right)\sin(3\tau) \end{aligned} \tag{2.38}$$

where $r_0 = \sqrt{a_0^2 + b_0^2} > 0$. So, for the periodic solution we have

$$\frac{1}{4}r_0^2(3a_0 - b_0) + 2\Omega_1 a_0 + b_0 = -F_1, \tag{2.39}$$

$$\frac{1}{4}r_0^2(3b_0 + a_0) + 2\Omega_1 b_0 - a_0 = 0, \tag{2.40}$$

together which implies the response function[84]

$$r_0^2\left(\frac{5}{8}r_0^4 + (3\Omega_1 - \frac{1}{2})r_0^2 + (4\Omega_1^2 + 1)\right) = F_1^2 \tag{2.41}$$

which give the possible amplitudes r_0 for given Ω_1 and F_1.

The Lindstedt–Poincaré perturbation theory is always a reliable technique in the region of small coupling constant. The harmonic balance result, on the other hand, if expanded in the perturbation parameter may not always lead to correct result.

Note that the Lindstedt–Poincaré method provides a way to construct asymptotic approximations of periodic solutions, but it cannot be used to obtain solutions that evolve aperiodically on a slow time-scale[253]. The method of multiple-scales is a more general approach that involves two key points. The first is the idea of introducing scaled space and time coordinates to capture the slow modulation of the pattern, and treating them as separate variables in addition to the original variables that must be retained to describe the patterned state itself. In the essential idea of multiple scales, an analytical solution, say $x(t;\epsilon)$ that the functional dependence of x on t and ϵ is not disjoint because x depends on the combination of ϵt as well as on the individual t and ϵ i.e. in place of $x = x(t;\epsilon)$ we write $x = x(t, \epsilon t; \epsilon)$.

2.8 Multiple time scales and renormalisation group (RG) technique

Multiple time scale analysis is an useful analytic tool for constructing uniform or global approximate solutions of independent variables. Asymptotic and perturbative analysis have played a significant role in applied science. When the regular perturbation techniques (as for example Reductive Perturbation Technique) are failed to show approximate result for a system, the singular perturbation technique are to be called for. Wentzel–Kramers–Brillouin (WKB) is a special case of multi-scale analysis, applicable on linear equations only whereas Krylov–Bogolyubov (K-B), Renormalisation Group (RG) etc. are such types of singular perturbation techniques which are dependent upon the multiple time scale analysis. These are most well known techniques and each of which has its particular drawback, preventing algorithmic application. The RG technique is the principal tool to calculate this universal behaviour and is properly regarded as a means of asymptotic analysis. In this analysis a set of scaled variables, which are regarded as independent variables although they are ultimately related to one another, is introduced to remove all secular terms. The essence of the RG method is to extract structurally stable features of a system which are insensitive to details.

Similarities between the RG and conventional singular perturbation method is that both removes the secular terms or divergent terms from the perturbation series from a periodic solution point of view. Chen et al.[46, 47] demonstrate that, singular perturbation methods may be understood as normalised perturbation theory and that amplitude equations obtainable by Reductive Perturbation Technique (RPT) may be derived as RG equations, also they indicate that the RG method may have several practical advantages compared with conventional methods. Recently it has been demonstrated to be an useful method for differentiating between center like oscillation and limit cycles for a two-dimensional dynamical system[48, 49, 166, 254–257].

Use of such techniques can serve in differentiating between oscillatory dynamics of a center-type or limit cycle. The center-type oscillation consists of a continuous family of closed orbits in phase space, each orbit being determined by its own initial condition.

2.8.1 *Renormalisation Group (RG) analysis for Liénard-type systems*

Here, the RG perturbative technique is given by stepwise algorithm from a general point of view for an autonomous nonlinear second order homogeneous ODE that contains a trivial fixed point surrounding by a periodic orbit i.e. a LLS system. As we are very much concerned about the perturbation theory for various types of nonlinear ODE, so, to hold the same, the system would have to be small enough otherwise the perturbative analysis will fail to provide the system characteristics or it might introduce error. So, whatever system or approaches are explained regarding this context, they are for weakly nonlinear systems. On the other hand, if any system does not have control parameter then perturbation theory tell us that a perturbation parameter (or nonlinearity control parameter, say, λ) has to be introduced artificially to perform perturbation theory. Finally one has to put $\lambda = 1$ to bring back to the original equation of motion.

Considering a general weakly nonlinear LLS system

$$\ddot{x} + \lambda F(x, \dot{x})\dot{x} + G(x) = 0, \tag{2.42}$$

with the initial values $x(t_0) = A$ and $\dot{x}(t_0) = 0$, where over dots represents the derivatives with respect to t and λ is a nonlinearity control parameter. The functions F and G satisfy the Liénard or LLS properties. For a linear restoring force, the natural frequency of the above system is ω and if there be a nonlinear restoring force then the frequency ω should have to be corrected that can be entered through phase equation. One can start the RG calculations by rescaling the natural frequency in the original time scale or with the natural frequency as it is. As we are talking about weak system, so λ lies between 0 and 1 i.e. $0 < \lambda \ll 1$. RG analysis for other types of two-dimensional or three-dimensional kinetic flow equations are well explained in [48, 49, 64, 254, 258, 259] which are not given here as we are discussing the perturbation theory here in the context of a LLS system.

So the basic steps of calculation are:

1. Consider a perturbative solution: $x(t) = x_0(t) + \lambda x_1(t) + O(\lambda^2)$.

2. Put the approximate solution, x, in (2.42).

3. Collect $\lambda^0, \ \lambda^1, \ldots$ from the both sides of (2.42) upto desired expectation.

4. Solve the first harmonics, $x_0(t)$ for λ^0 by using initial values.

5. Solve the second harmonics, $x_1(t)$ for λ^1, by using $x_0(t)$ along with initial values.
   ...repeat the last step n-times to solve the higher order harmonics for λ^n.

6. Write the solution of $x(t)$ in terms of $x_0(t), x_1(t), \ldots$ i.e. $x(t) = x_0(t) + \lambda x_1(t) + O(\lambda^2)$. It has a constant amplitude, A, and phase, $\theta_0 = -\omega t_0$; ω is the system frequency in absence of nonlinearity.

7. Take a perturbation in the time interval $(t - t_0)$ by splitting the interval $(t - t_0) = (t - \tau) + (\tau - t_0)$, $t_0 < \tau \ll t$ and τ is very close to t_0; the interval $(t - \tau)$ is called the principal part and the non principal part (i.e. $\tau - t_0$) can be neglected because of smallness.

8. If any term is multiplied directly by $(t - t_0)$ in the final solution of $x(t)$, then convert it into $(t - \tau)$ by neglecting the other part.

9. Two renormalisation constants $(Z_1(\tau, t_0), Z_2(\tau, t_0))$ have to be introduced to absorb the divergence owing to the secular terms, $\cos(\omega t + \theta_0)$, $\sin(\omega t + \theta_0)-$ are the first oscillations, where $A(\tau) = \frac{A}{Z_1(\tau,t_0)}$ and $\theta(\tau) = \theta_0 - Z_2(\tau, t_0)$ i.e. $Z_1(\tau, t_0)$ and $Z_2(\tau, t_0)$ are the multiplicative and additive perturbations to the amplitude and phase, respectively. They are introduced to know the stability of amplitude and phase. The series of $Z_1(\tau, t_0)$ and $Z_2(\tau, t_0)$ are of the form,

$$\begin{aligned} Z_1(\tau, t_0) &= 1 + \sum_1^\infty \lambda^n p_n = 1 + \lambda p_1 + O(\lambda^2), \\ Z_2(\tau, t_0) &= 0 + \sum_1^\infty \lambda^n q_n = \lambda q_1 + O(\lambda^2). \end{aligned} \tag{2.43}$$

10. Put A and θ_0 in the final solution and remove the terms which could lead to divergence. This will provide (p_1, q_1) as a function of $(\tau - t_0)$ which are responsible for occurring the divergence, and one can ignore them as $(\tau - t_0)$ is very small.

11. The last two steps effectively implies that the amplitude and phase are slightly changed from (A, θ_0) to $(A(\tau), \theta(\tau))$ in the solution of $x(t)$ ($\approx x(t, \tau)$—in the step 6) i.e. they are weakly time dependent due to introducing the arbitrary time scale variable τ.

12. The final solution, $x(t, \tau)$, cannot depend on the arbitrary time scale, τ, i.e. $\left(\frac{\partial x}{\partial \tau}\right)_t = 0$, which leads to

$$\begin{aligned} \frac{dA}{d\tau} &= f(A(\tau)), \\ \frac{d\theta}{d\tau} &= g(A(\tau)). \end{aligned} \tag{2.44}$$

13. The radius of the cycle can be obtained by setting $f(A) = 0$ if there exist any $A \neq 0$ and if all A are trivial for $f(A) = 0$ then we may call this as a center or center-type oscillation.

14. The independence of θ upon τ, i.e. $\frac{d\theta}{d\tau} = g(A) = 0$ gives the condition for isochronicity i.e. if $g(A) \neq 0$ then the Liénard system would not be isochronous.

In a recent communication Das et al.[255, 256] showed that the above rigorous calculation can be fixed just by looking at the solution of $x_1(t)$ which is able to provide the amplitude and phase equations as it does contain the secular terms for the first order correction only. There is no need to calculate further after the 5^{th} step. But it is better to follow all the steps otherwise it would not be easy to find the connection between the amplitude and phase equations with the approximate solution.

Example 2.8.1. As for example choosing the same Van der Pol–Duffing system case,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x - \lambda x^3 = 0, \tag{2.45}$$

having the initial values $x(t_0) = A$ and $\dot{x}(t_0) = 0$ and taking a special situation, $\lambda = \epsilon$, and consider a perturbative solution of $x(t)$ as $x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2)$. Then after substituting into Eq. (2.45) we have the simplified form,

$$(\ddot{x}_0 + \epsilon\ddot{x}_1) + \epsilon(x_0^2 - 1)\dot{x}_0 + (x_0 + \epsilon x_1) - \epsilon x_0^3 + O(\epsilon^2) = 0, \tag{2.46}$$

Then Eq. (2.46) gives (after neglecting $O(\epsilon^2)$),

$$\epsilon^0 : \quad \ddot{x}_0 + x_0 \quad = \quad 0, \tag{2.47}$$
$$\epsilon^1 : \quad \ddot{x}_1 + x_1 \quad = \quad -(x_0^2 - 1)\dot{x}_0 + x_0^3. \tag{2.48}$$

Now, initial values $(x_0(t_0) = A,\ \dot{x}_0(t_0) = 0)$ gives the solution of the zeroth order i.e. x_0 is $x_0 = A\cos(t + \theta_0),\ -t_0 = \theta_0$. Then the first order equation takes the form,

$$\ddot{x}_1 + x_1 = -(A^2\cos^2(t + \theta_0) - 1)(-A\sin(t + \theta_0)) + A^3\cos^3(t + \theta_0). \tag{2.49}$$

By using initial values ($x_1(t_0) = 0,\ \dot{x}_1(t_0) = 0$), solution of the above equation takes the form,

$$
\begin{aligned}
x_1 = {} & \frac{7}{32}A^3 \sin(\theta_0 + t) + \frac{3}{8}A^3 t \sin(\theta_0 + t) - \frac{1}{32}A^3 \sin(3\theta_0 + 3t) + \frac{3}{8}A^3\theta_0 \sin(\theta_0 + t) \\
& + \frac{1}{32}A^3 \cos(\theta_0 + t) - \frac{1}{8}A^3 t \cos(\theta_0 + t) - \frac{1}{32}A^3 \cos(3\theta_0 + 3t) \\
& - \frac{1}{8}A^3\theta_0 \cos(\theta_0 + t) - \frac{1}{2}A \sin(\theta_0 + t) + \frac{1}{2}At \cos(\theta_0 + t) + \frac{1}{2}A\theta_0 \cos(\theta_0 + t)\,.
\end{aligned} \tag{2.50}
$$

Then a complete perturbative solution, $x(t) = A\cos(t + \theta_0) + \epsilon x_1(t) + O(\epsilon^2)$ becomes (after applying step 11)

$$
\begin{aligned}
x(t) = {} & A(\tau)\cos(\theta(\tau) + t) + \\
& \epsilon\left(\frac{7}{32}A(\tau)^3 \sin(\theta(\tau) + t) + \frac{3}{8}tA(\tau)^3 \sin(\theta(\tau) + t) - \frac{1}{32}A(\tau)^3 \sin(3\theta(\tau) + 3t)\right. \\
& + \frac{3}{8}A(\tau)^3\theta(\tau)\sin(\theta(\tau) + t) - \frac{1}{2}A(\tau)\sin(\theta(\tau) + t) + \frac{1}{32}A(\tau)^3 \cos(\theta(\tau) + t) \\
& - \frac{1}{8}tA(\tau)^3 \cos(\theta(\tau) + t) - \frac{1}{32}A(\tau)^3 \cos(3\theta(\tau) + 3t) - \frac{1}{8}A(\tau)^3\theta(\tau)\cos(\theta(\tau) + t) \\
& \left. + \frac{1}{2}tA(\tau)\cos(\theta(\tau) + t) + \frac{1}{2}A(\tau)\theta(\tau)\cos(\theta(\tau) + t)\right) + O(\epsilon^2).
\end{aligned} \tag{2.51}
$$

Now introducing an arbitrary small time, τ through scaling, $(t - t_0) = (t - \tau) + (\tau - t_0)$ and neglecting the non-principal part we have,

$$
\begin{aligned}
x(t, \tau) = {} & A(\tau)\cos(\theta(\tau) + t) + \\
& \epsilon\left(\frac{7}{32}A(\tau)^3 \sin(\theta(\tau) + t) + \frac{3}{8}(t - \tau)A(\tau)^3 \sin(\theta(\tau) + t) - \frac{1}{32}A(\tau)^3 \sin(3\theta(\tau) + 3t)\right. \\
& + \frac{3}{8}A(\tau)^3\theta(\tau)\sin(\theta(\tau) + t) - \frac{1}{2}A(\tau)\sin(\theta(\tau) + t) + \frac{1}{32}A(\tau)^3 \cos(\theta(\tau) + t) \\
& - \frac{1}{8}(t - \tau)A(\tau)^3 \cos(\theta(\tau) + t) - \frac{1}{32}A(\tau)^3 \cos(3\theta(\tau) + 3t) - \frac{1}{8}A(\tau)^3\theta(\tau)\cos(\theta(\tau) + t) \\
& \left. + \frac{1}{2}(t - \tau)A(\tau)\cos(\theta(\tau) + t) + \frac{1}{2}A(\tau)\theta(\tau)\cos(\theta(\tau) + t)\right) + O(\epsilon^2).
\end{aligned} \tag{2.52}
$$

The above solution is obtained after removing all the secular terms in (2.51) by $p_1 = -\frac{1}{8}A(A^2 - 4)(\tau - t_0)$ and $q_1 = -\frac{3}{8}A^2(\tau - t_0)$. Now applying the final step i.e. the final solution cannot be dependent on the arbitrary time scale, τ, i.e. $\left(\frac{\partial x}{\partial \tau}\right)_t = 0$, which leads to,

$$
\begin{aligned}
\frac{dA}{d\tau} &= -\frac{1}{8}A(\tau)(A^2(\tau) - 4), \\
\frac{d\theta}{d\tau} &= -\frac{3}{8}\epsilon A^2(\tau).
\end{aligned} \tag{2.53}
$$

This implies the system has a limit cycle of approximate amplitude 2 along with the frequency correction of amount $-\frac{3}{8}\epsilon A^2(\tau)$ due to the bistable potential. Thus the value of the amplitude is obtained as the distance from the origin (i.e., the fixed point) and the system will not be an isochronous oscillator as θ decreases slowly with time.

2.8.2 *Krylov–Bogolyubov (K-B) method*

Let us consider the form of a weakly nonlinear oscillator as,

$$\ddot{x}(t) + \epsilon h(x(t), \dot{x}(t)) + \omega^2 x(t) = 0,\ \ 0 < \epsilon \ll 1, \tag{2.54}$$

where, $h(x, \dot{x})$ is a function of the position variable, x and the velocity variable, $\dot{x}$ which contains nonlinearity coming from nonlinear damping or restoring force. The nonlinearity control parameter, ϵ, must be within 0 and 1. Now, let us rewrite Eq. (2.54) in the form as,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x - \epsilon h(x, \dot{x}). \end{aligned} \tag{2.55}$$

For $\epsilon = 0$, the above system reduces to simple harmonic oscillator (SHO) with natural frequency ω having a solution, $x(t) = r\cos(\omega t + \phi)$ and $y(t) = -\omega\ r\sin(\omega t + \phi)$, with constant amplitude, $r = \sqrt{x^2 + \frac{y^2}{\omega^2}}$ and phase, $\phi = -\omega t + \tan^{-1}(-\frac{y}{\omega x})$ having the circular orbit of period $\frac{2\pi}{\omega}$.

For $\epsilon \neq 0$ i.e. the nonlinear terms are opened and the vector field changes by an amount ϵ, and for small ϵ, all orbits are nearly circular which approximately repeat in every $\frac{2\pi}{\omega}$, and because any value introduces a tiny change, so one can really change the period by more than some amount proportional to ϵ. Then two questions may arise, how to find a limit cycle and what will be the amplitude of the cycle?

In what follows, the method is the adoption of the combination of power consumption law along with the averaging theory[10]. For a SHO, the total energy is conserved and as we go around one cycle the energy has not changed. The system with ϵ, sometimes getting pumped and sometimes damped and that's why it is an interesting term. When it repeats in the system, the limit cycle occurs because of the mutual effect of them. So, making the solutions be non-autonomous i.e. the amplitude, r and phase, ϕ being explicitly time dependent one can get an idea about the evolution of the solutions for weak nonlinearity.

Therefore, considering $x(t) \approx r(t)\cos(\omega t + \phi(t))$ and $y(t) \approx -\omega\ r(t)\sin(\omega t + \phi(t))$ with $r(t) \approx \sqrt{x^2(t) + \frac{y^2(t)}{\omega^2}}$ and $\phi(t) \approx -\omega t + \tan^{-1}\left(-\frac{y(t)}{\omega x(t)}\right)$, then one can have the rate of change of amplitude and phase variables with respect to time as,

$$
\begin{aligned}
\dot{r}(t) &= \frac{\epsilon\ h}{\omega}\sin(\omega t + \phi(t)),\\
\dot{\phi}(t) &= \frac{\epsilon\ h}{\omega\ r(t)}\cos(\omega t + \phi(t)).
\end{aligned} \tag{2.56}
$$

This implies that the time derivatives of amplitude and phase are of $O(\epsilon)$. Now, if $\overline{U}(t)$ be a running average of a time dependent function U defined as,

$$
\overline{U}(t) = \frac{\omega}{2\pi}\int_{t-\frac{\pi}{\omega}}^{t+\frac{\pi}{\omega}} U(s)\ ds \qquad \text{or} \qquad \overline{U}(t) = \frac{\omega}{2\pi}\int_{0}^{\frac{2\pi}{\omega}} U(s)\ ds, \tag{2.57}
$$

then from the fundamental theorem of calculus it is observed that $\dot{\overline{U}} = \overline{\dot{U}}$. By applying this averaging trick to the time dependent amplitude and phase equations for each cycle (as the trial solution is taken approximately periodic) then we have,

$$
\begin{aligned}
\dot{\overline{r}} &= \langle \tfrac{\epsilon\ h(x,y)}{\omega}\ \sin(\omega t + \phi(t))\rangle_t,\\
\dot{\overline{\phi}} &= \langle \tfrac{\epsilon\ h(x,y)}{\omega\ r(t)}\cos(\omega t + \phi(t))\rangle_t.
\end{aligned} \tag{2.58}
$$

As $\dot{r}(t)$ and $\dot{\phi}(t)$ are of $O(\epsilon)$ then we may set the perturbation on r and ϕ over each cycle as,

$$
\begin{aligned}
r(t) &= \overline{r} + O(\epsilon),\\
\phi(t) &= \overline{\phi} + O(\epsilon),
\end{aligned} \tag{2.59}
$$

where, $\overline{r}$ and $\overline{\phi}$ are very weakly time dependent so that the error can be negligible. Finally, from the above consideration, one can obtain,

$$
\begin{aligned}
\dot{\overline{r}} &= \langle \frac{\epsilon\ h(\overline{r}\cos(\omega t + \overline{\phi}), -\omega\ \overline{r}\sin(\omega t + \overline{\phi})}{\omega}\ \sin(\omega t + \overline{\phi})\rangle_t = \varphi_1(\overline{r}, \overline{\phi}),\\
\dot{\overline{\phi}} &= \langle \frac{\epsilon\ h(\overline{r}\cos(\omega t + \overline{\phi}), -\omega\ \overline{r}\sin(\omega t + \overline{\phi})}{\omega\ \overline{r}}\cos(\omega t + \overline{\phi})\rangle_t = \varphi_2(\overline{r}, \overline{\phi}),
\end{aligned} \tag{2.60}
$$

where, $O(\epsilon^2)$ terms can be neglected as first order approximation is considered here. The above systems will not be in a coupled form for autonomous set of equation, but for the non-autonomous case we may have coupling between amplitude and phase flow variables in the flow equation and that kind of situation may provide very complex phenomena which is quite harder to solve as well as to get back the original system of equations which can be fixed by considering further approximations.

Example 2.8.2. For an example, let us choose the Van der Pol–Duffing system,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x - \lambda x^3 = 0, \tag{2.61}$$

having the initial values $x(t_0) = A$ and $\dot{x}(t_0) = 0$. Now, considering a special situation, $\lambda = \epsilon$, the above system can be written as,

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0;\; h(x, \dot{x}) = (x^2 - 1)\dot{x} - x^3. \tag{2.62}$$

So, for $\epsilon = 0$, the harmonic oscillator solution will be $x(t) = r\cos(t + \phi)$ and $y(t) = \dot{x}(t) = -r\sin(t + \phi)$ with $(r, \phi) = (\sqrt{x^2 + y^2}, -t + \tan^{-1}(-\frac{y}{x}))$. For $\epsilon \neq 0$, the solution will be slightly modified to $x(t) \approx r(t)\cos(t + \phi(t))$ and $y(t) \approx -r(t)\sin(t + \phi(t))$ with

$$(r(t), \phi(t)) \approx \left(\sqrt{x^2(t) + y^2(t)}, -t + \tan^{-1}\left(-\frac{y(t)}{x(t)}\right)\right).$$

Then amplitude and phase equation (before averaging) becomes,

$$\begin{aligned} \dot{r}(t) &= \epsilon\, h(x, \dot{x}) \sin(t + \phi(t)), \\ \dot{\phi}(t) &= \frac{\epsilon\, h(x, \dot{x})}{r(t)} \cos(t + \phi(t)), \end{aligned} \tag{2.63}$$

with $r(t) = \overline{r} + O(\epsilon)$ and $\phi(t) = \overline{\phi} + O(\epsilon)$, where $\overline{r}$ and $\overline{\phi}$ are very weakly time dependent. Then the average amplitude and phase equations are of the form (after neglecting $O(\epsilon^2)$ terms),

$$\begin{aligned} \dot{\overline{r}} &= \langle \epsilon\, h(\overline{r}\cos(t + \overline{\phi}), -\overline{r}\sin(t + \overline{\phi}))\; \sin(t + \overline{\phi}) \rangle_t, \\ \dot{\overline{\phi}} &= \langle \frac{\epsilon\, h(\overline{r}\cos(t + \overline{\phi}), -\overline{r}\sin(t + \overline{\phi}))}{\overline{r}} \cos(t + \overline{\phi}) \rangle_t, \end{aligned} \tag{2.64}$$

with

$$\begin{aligned} h &= (x^2 - 1)\dot{x} - x^3 \\ &= -\overline{r}\,(\overline{r}^2\cos^2(t + \overline{\phi}) - 1)\; \sin(t + \overline{\phi}) - \overline{r}^3\cos^3(t + \overline{\phi}) + O(\epsilon). \end{aligned} \tag{2.65}$$

Finally, from the above consideration, integrating we have,

$$\begin{aligned} \dot{\overline{r}} &= \frac{\epsilon}{2}\left(\overline{r} - \frac{\overline{r}^3}{4}\right), \\ \dot{\overline{\phi}} &= -\frac{3\,\epsilon}{8}\overline{r}^3. \end{aligned} \tag{2.66}$$

For the considered autonomous system the amplitude equation is completely dependent on $\overline{r}$ (independent of $\overline{\phi}$) but the phase equation is dependent upon $\overline{r}$. One can have the phase correction once the amplitude is known. The amplitude equation shows that the system has a limit cycle with radius of magnitude 2.

The existence of a limit cycle (or a periodic attractor) can be predicted by the Poincaré–Bendixson theorem, but the exact location and size of the limit cycle cannot be predicted in advance. This is why perturbation theory is required to arrive at some amplitude equation whose fixed point are supposedly give us the location of the limit cycle, if exist. This amplitude equation helps us to understand more quantitatively about the location of the limit cycle or how large it will be. But, if one is interested in knowing finer details about the shape and size of the limit cycles for these types of LLS oscillators, the procedure of approaching the problem perturbatively becomes non-trivial. So perturbation theory (up to a reliable order) provides us flow equations in amplitude and phase and their fixed points tell us where the system will ultimately settle. For example, to first order in perturbation, the amplitude of the Van der Pol oscillator ($\lambda = 0$ in 2.61) gives a limit cycle and for the Duffing oscillator ($\epsilon = 0$ in 2.61) the frequency needs to be corrected with no change in amplitude. As it turns out, first order perturbation theory gives us very practicable and accurate results, independent of which technique is used to arrive at them, viz., multiple time scale steps[34, 53], generalized averaging steps[34, 44, 54] or, of more recent practice of the RG method steps[46, 48, 49].

2.9 Approximate solution of nonlinear oscillator

It is very difficult to find an exact analytical solution of a nonlinear system except only for a very few special cases but for most of the systems exact analytical solution is almost impossible. Sometimes perturbative approach is able to provide an approximate analytical solution of a nonlinear system and the exactness of the solution will depend upon the coefficient of the nonlinearity i.e. the solutions are more exact as much as the system contains weak nonlinearity. Then one can predict an approximate analytical solution in regard to the multi-scale perturbative approach to obtain the amplitude and phase equation.

2.9.1 *Approximate solution by Renormalisation Group (RG)*

Let us describe more elaborately starting from the amplitude and phase equation obtained from RG approach (Eq. 2.53). If we integrate the amplitude separately then we have

$$A(\tau) = \frac{2e^{\frac{\tau\epsilon}{2}}}{\sqrt{c + e^{\tau\epsilon}}}$$

and the phase solution will be

$$\theta(\tau) = d - \frac{3}{8}A(\tau)^2\tau\epsilon$$

where c and d are the integrating constants which can be fixed by the initial conditions. Now, once we obtain the amplitude and phase then we can put it in the solution of $x(t)$ in (2.51) where the final removal of the τ-dependence can be done through the choice $\tau = t$. This will provide an approximate analytical solution in the form

$$\begin{aligned} x(t) = {} & A(t)\cos(\theta(t)+t) + \\ & \epsilon\left(\frac{7}{32}A(t)^3\sin(\theta(t)+t) + \frac{3}{8}tA(t)^3\sin(\theta(t)+t)\right. \\ & - \frac{1}{32}A(t)^3\sin(3\theta(t)+3t) + \frac{3}{8}A(t)^3\theta(t)\sin(\theta(t)+t) - \frac{1}{2}A(t)\sin(\theta(t)+t) \\ & + \frac{1}{32}A(t)^3\cos(\theta(t)+t) - \frac{1}{8}tA(t)^3\cos(\theta(t)+t) - \frac{1}{32}A(t)^3\cos(3\theta(t)+3t) \\ & \left. - \frac{1}{8}A(t)^3\theta(t)\cos(\theta(t)+t) + \frac{1}{2}tA(t)\cos(\theta(t)+t) + \frac{1}{2}A(t)\theta(t)\cos(\theta(t)+t)\right) + O(\epsilon^2), \end{aligned} \tag{2.67}$$

with

$$\begin{aligned} A(t) &= \frac{2e^{\frac{t\epsilon}{2}}}{\sqrt{c+e^{t\epsilon}}} \text{ and} \\ \theta(t) &= d - \frac{3}{8}A(t)^2 t\epsilon. \end{aligned} \tag{2.68}$$

Moreover, total energy (approximated) of the considered Van der Pol—Duffing system can directly be calculated by the formula given by

$$E = \frac{\dot{x}^2}{2} + \left[\frac{x^2}{2} - \lambda\,\frac{x^4}{4}\right], \tag{2.69}$$

once the solution of $x(t)$ is known and the period of the oscillation can be calculated in terms of an elliptic integral with

$$T = 2\int_{-A}^{A} dx\,\frac{1}{\sqrt{2(E-V(x))}},\quad V(x) = \left[\frac{x^2}{2} - \lambda\,\frac{x^4}{4}\right] \tag{2.70}$$

where A is the amplitude of the oscillation.

2.9.2 *Approximate solution by Krylov–Bogolyubov (K-B)*

In addition to the above consideration, if we want to find an approximate solution by using K-B perturbative method we have to calculate the similar integral for the amplitude and then we can calculate the phase solution. So, from (2.66), we have

$$\begin{aligned} \overline{r} &= \frac{2e^{\frac{t\epsilon}{2}}}{\sqrt{c+e^{t\epsilon}}} \text{ and} \\ \overline{\phi} &= d - \frac{3}{8}\overline{r}^2 t\epsilon, \end{aligned} \tag{2.71}$$

where c and d are the integrating constants which can be fixed by the initial conditions by using r_0 and ϕ_0 i.e. $x(t_0)$ and $\dot{x}(t_0)$. Therefore, a general approximate phase space solution upto the first order of the considered Van der Pol—Duffing system are in the form,

$$\begin{aligned} x(t) &= \overline{r}\cos\left(t+\overline{\phi}\right) + O(\epsilon), \\ y(t) &= -\overline{r}\sin\left(t+\overline{\phi}\right) + O(\epsilon), \end{aligned} \tag{2.72}$$

where, $\overline{r}$ and $\overline{\phi}$ are given above.

The approximate energy becomes of the same form as in the undamped case due to the weak nonlinearity (i.e. $0 < \epsilon \ll 1$), which is, $E = \frac{1}{2}(x^2 + \dot{x}^2)$. The correctness due to the error can be neglected as ϵ is very small. So the change in energy or energy consumption over each cycle is,

$$\Delta E = \int_0^T \frac{dE}{dt} dt = \int_0^{2\pi+O(\epsilon)} \frac{dE}{dt} dt = 2\pi\, \overline{r}\, \dot{\overline{r}} + O(\epsilon^2) = \frac{d}{dt}\left(\pi\overline{r}^2\right) + O(\epsilon^2). \tag{2.73}$$

2.10 VAN DER POL—DUFFING IN THE FORM OF $\lambda - \omega$

Here, we have shown the $\lambda - \omega$ type kinetic form of Van der Pol–Duffing system

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x - \epsilon x^3 = 0. \tag{2.74}$$

The above system has a unrefined kinetic set of equations, say, $\dot{x} = y,\ \dot{y} = -\epsilon(x^2 - 1)\dot{x} - x + \epsilon x^3$, that does not correspond to a $\lambda - \omega$ or $\lambda - \omega$ type system. Then the point is, what will be its $\lambda - \omega$ type kinetic form, if any?

To answer this precise question recalling the $\lambda - \omega$ system having the following kinetic form:

$$\begin{aligned}\frac{dx}{dt} &= \lambda(r)x - \omega(r)y,\\ \frac{dy}{dt} &= \omega(r)x + \lambda(r)y; \quad r = \sqrt{x^2 + y^2}.\end{aligned}$$

one can have the amplitude and phase equations in polar coordinate,

$$\begin{aligned}\frac{dr}{dt} &= r\lambda(r),\\ \frac{d\phi}{dt} &= \omega(r).\end{aligned}$$

Now, applying K-B perturbative method as given in section 2.8.2 to the Van der Pol–Duffing system (2.74) we have the following amplitude and phase equations:

$$\begin{aligned}\dot{\bar{r}} &= \frac{\epsilon}{2}\left(\bar{r} - \frac{\bar{r}^3}{4}\right),\\ \dot{\bar{\phi}} &= -\frac{3\,\epsilon}{8}\bar{r}^3.\end{aligned}$$

Then let us rewrite the above equations into the form as,

$$\begin{aligned}\dot{\bar{r}} &= \bar{r}\frac{\epsilon}{8}\left(4 - \bar{r}^2\right) = \bar{r}\lambda(\bar{r}); \quad & \lambda(\bar{r}) &= \frac{\epsilon}{8}\left(4 - \bar{r}^2\right),\\ \dot{\bar{\phi}} &= \omega(\bar{r}); & \omega(\bar{r}) &= -\frac{3\,\epsilon}{8}\bar{r}^3.\end{aligned} \tag{2.75}$$

Now, if we write a two-dimensional kinetic $\lambda - \omega$ system with the above forms of $\lambda(r)$ and $\omega(r)$ then the system looks like,

$$\begin{aligned}\frac{dx}{dt} &= \frac{\epsilon}{8}\left(4 - \bar{r}^2\right)x - \left(-\frac{3\,\epsilon}{8}\bar{r}^3\right)y,\\ \frac{dy}{dt} &= \left(-\frac{3\,\epsilon}{8}\bar{r}^3\right)x + \frac{\epsilon}{8}\left(4 - \bar{r}^2\right)y; \quad \bar{r} = \sqrt{x^2 + y^2}.\end{aligned} \tag{2.76}$$

Simulation of the above equation agrees with Eq. (2.74) which has a limit cycle of radius ≈ 2 (like Van der Pol). There is a structural difference between them, like the cycle for the $\lambda - \omega$ form of the above system is circular whereas the structure of the cycle of system (2.74) is not exactly circular. As a conclusion one can say that weaker the nonlinearity a system will provide better agreement as orbits of such weakly Liénard system (Eq. 2.74) shows more circular in nature (closer to harmonic oscillator solution). More details are given in Appendix A.

Over the last few decades there are tremendous progress made in the area of dissipative dynamical systems but the problem is still there to locate the periodic orbits and its nature of a two-dimensional system[243]. The periodicity of orbits may come in two varieties–one is limit cycles and the other is center or center-type oscillations. The center or center-type orbits are the continuous family of closed curves in phase space dependent upon a prescribed initial condition but the limit cycle orbit is an isolated periodic trajectory where the nearby trajectories are attracted by the closed orbit or move far away from the isolated orbit. A little more than two decades ago Chen et al.[46, 47] proposed a different way of looking at the problem of nonlinear dynamical systems of oscillators which has been explored by several groups. This method involves a direct use of perturbation theory and RG approach. The RG naturally leads to flow equations. In this respect it is akin to the K-B method. The advantage, however, lies in the fact that RG uses naíve perturbation theory. One does not need to anticipate scales (as in multiple scales method) or make an assumption about slowly varying amplitude and phase (Krylov–Bogoliubov method).

Now a question may arise regarding the applicability of the multi-scale perturbative methods like K-B, Lindstedt-Poincaré, RG etc. Except the RG, other methods are related to the problems since 18th century and people have tried to understand through various simple as well as complicated systems[10, 34, 54, 57] by considering a harmonic solution in the weak limit of the system. In a recent development by Sarkar et al.[260] have also tried to understand the application of the RG principle through problems in dynamics for various 2-D systems where they begin by observing that a periodic solution can be expressed as a Fourier series with amplitude A and phase θ of the lowest harmonic which is determining the amplitude and phase of the higher order ones in the flow equations. A simple perturbative series expansion of the dynamical variable may lead to a divergent answer. If the time 't', at which $x(t)$ is desirable and by the help of the initial time 't_0', then $x(t)$ will diverge as $t - t_0 \to \infty$ and this is completely similar to divergence in field theory. As, the discussion is about a physical variable, then the answer has to be finite and while this is achieved in field theory by constructing running coupling constants which is done for the differential equation by introducing an arbitrary time scale τ and letting the amplitude and phase depend on τ. Further development of it Das et al.[255, 256] have modified the RG scheme so that one can not go through a rigorous calculation after finding a full series solution of $x(t)$

where the amplitude and the phase equations can be predictable by the second harmonics. After applying all the methodology one can have the the flow equations.

$$\frac{dA}{d\tau} = f(A, \theta), \tag{2.77a}$$

$$\frac{d\theta}{d\tau} = g(A, \theta). \tag{2.77b}$$

For the autonomous system of f and g are generally function of A only. We propose to use the above flow equations to differentiate between oscillators which are of the center variety and limit cycles. The slowly decaying center-type oscillation consists of a continuous family of closed orbits in phase space where each orbit being determined by its own initial condition. This implies that the amplitude A are fixed, once the initial condition is set. This must lead to

$$\frac{dA}{d\tau} = 0. \tag{2.78}$$

This statement is exact and is not tied to any perturbation theory argument. For the limit cycle on the other hand the condition is

$$\frac{dA}{d\tau} = f(A) \tag{2.79}$$

and $f(A)$ must be such that the flow must have a fixed point. The fixed point of $f(A)$ has to be stable for a stable limit cycle. If one finds $A = 0$ which is the only fixed point of $f(A)$, then it is a focus.

3

ISOCHRONICITY AND LIMIT CYCLE OSCILLATION IN CHEMICAL SYSTEMS

3.1 INTRODUCTION

The generic features of diverse nature of nonlinear chemical oscillations are due to autocatalysis and various feedback mechanisms into the system which are basically controlled by a few slowest time scales of the overall process. The coupled dynamics of the system can be described by two intermediate concentrations or population variables characterized by the occurrence of a limit cycle when the motion is visualized on a phase plane. In particular, a procedure of the reduction of chemical cubic equations to the form of a second order differential equation with coefficients which allows for the limit cycle analysis so called Rayleigh oscillator has been proposed for a first time in the article by Lavrova et al.[15, 22]. Its further development to a more general case, the Liénard oscillator was given by Ghosh and Ray[40]. Here we consider that a class of arbitrary, autonomous kinetic equations in two variables describing chemical oscillations can be cast into the form of a Liénard oscillator[34, 40, 48, 49, 58, 64]. It is characterized by the nonlinear forcing and damping coefficients which can control the limit cycle behaviour.

Although nonlinear oscillators got a lot of attention over the years in the field of dynamical systems but there is no straightforward way of distinguishing between limit cycle and isochronous orbit of the system with their very different kinds of solutions. Here our effort is to study the isochronous systems by analyzing the behaviour around a center by Renormalisation Group (RG) method[46, 47] and to find the so called periodic orbits i.e. conditions under which a system becomes isochronous[48, 49]. Our method of distinguishing center and limit cycle behaviours are numerically analyzed here in terms of the parameters of the chemical oscillators. When the two-dimensional kinetic equations are transformed into a Liénard system of equation we would like to find here the relation between the limit cycle and isochronicity in an open dynamical system. As the chemical oscillators are standard real experimental models the theory is verified here in various systems.

In section (3.2), we have briefly reviewed the method of reduction of kinetic equation into Liénard form to find the condition for limit cycle. Isochronicity for Liénard System is described in section (3.3). In section (3.4), we have shown the examples with (3.4.1) for modified Brusselator model, (3.4.2) for Glycolytic oscillator and (3.4.3) for Van der Pol type oscillator to analyze the behaviour of limit cycle and isochronicity. The chapter[1] is concluded in section (3.5).

3.2 REDUCTION OF KINETIC EQUATION INTO LIÉNARD FORM: CONDITIONS FOR LIMIT CYCLE

Let us consider a two-dimensional set of autonomous kinetic equations for open system. Here our purpose is to review the condition for limit cycle by casting the two dynamical equations into a form of Liénard oscillator[10, 34, 40, 48, 58, 64]. Following the analysis of Ghosh and Ray[40] we consider a system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= a_0 + a_1 x + a_2 y + f(x, y), \\ \frac{dy}{dt} &= b_0 + b_1 x + b_2 y + g(x, y), \end{aligned} \tag{3.1}$$

where x and y are populations of two intermediate species of a dynamical process with $a_0, a_1, a_2, b_0, b_1, b_2$ are all real parameters expressed in terms of the kinetic constants with $f(x, y)$ and $g(x, y)$ are nonlinear functions.

Then writing the equations in terms of a new pair, (z, u) as

$$\begin{aligned} z &= \beta_0 + \beta_1 x + \beta_2 y, \\ u &= \alpha_0 + \alpha_1 x + \alpha_2 y, \end{aligned} \tag{3.2}$$

where $\alpha_0, \alpha_1, \alpha_2$ and $\beta_0, \beta_1, \beta_2$ are constants expressed in terms of a_i and b_i.

From the inverse transform of the above we can easily obtain the expressions of x and y as given by

$$\begin{aligned} x &= \frac{\beta_2(u - \alpha_0) - \alpha_2(z - \beta_0)}{\alpha_1\beta_2 - \beta_1\alpha_2} = L(u, z), \\ y &= \frac{\beta_1(u - \alpha_0) - \alpha_1(z - \beta_0)}{\alpha_2\beta_1 - \beta_2\alpha_1} = M(u, z). \end{aligned}$$

Choosing u and z in such a way that,

$$\frac{dz}{dt} = u, \tag{3.3}$$

1 Some portion of this chapter is published in the J. Math. Chem. - Saha et al. (2017)

and differentiating again w.r.t. t, we get

$$\begin{aligned}\ddot{z} &= \dot{u} = \alpha_1\dot{x} + \alpha_2\dot{y} \\ &= \alpha_1\{a_0 + a_1 L(z,\dot{z}) + a_2 M(z,\dot{z}) + \varphi(z,\dot{z})\} + \alpha_2\{b_0 + b_1 L(z,\dot{z}) + b_2 M(z,\dot{z}) + \phi(z,\dot{z})\}\end{aligned} \tag{3.4}$$

where

$$L(z,\dot{z}) = c_1 z + c_2\dot{z} + c_L$$

with

$$c_1 = -\frac{\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1},\ c_2 = \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1},\ c_L = \frac{\alpha_2\beta_0 - \alpha_0\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}$$

and

$$M(z,\dot{z}) = c_3 z + c_4\dot{z} + c_M$$

with

$$c_3 = -\frac{\alpha_1}{\alpha_2\beta_1 - \alpha_1\beta_2},\ c_4 = \frac{\beta_1}{\alpha_2\beta_1 - \alpha_1\beta_2},\ c_M = \frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_2\beta_1 - \alpha_1\beta_2}.$$

Next we consider c_L and c_M to be negligibly small. It is trivial to assume that both the numerators will not exactly vanish as $\alpha_2\beta_0 = \alpha_0\beta_2$ and $\alpha_0\beta_1 = \alpha_1\beta_0$ together giving $\alpha_2\beta_1 = \alpha_1\beta_2$ which makes all constants c_i and the system to be undefined. Here it is performed by choosing the ratio of numerator and denominator for the constants c_L and c_M are very small. Subsequently we define $L(z,\dot{z})$ and $M(z,\dot{z})$ by ignoring the small values of c_L and c_M, respectively.

Now taking the functions φ and ϕ as power series i.e.

$$\begin{aligned}\varphi(z,\dot{z}) &= \sum_{n,m=0}^{\infty} \varphi_{nm} z^n \dot{z}^m, \\ \phi(z,\dot{z}) &= \sum_{n,m=0}^{\infty} \phi_{nm} z^n \dot{z}^m\end{aligned} \tag{3.5}$$

in Eq. (3.5) one finds

$$\ddot{z} = A_{00} + A_{10}z + A_{01}\dot{z} + \sum_{n>1}^{\infty} A_{n0}z^n + \sum_{m>1}^{\infty} A_{0m}\dot{z}^m + \sum_{n,m\geqslant 1}^{\infty} A_{nm}z^n\dot{z}^m, \tag{3.6}$$

where, $A_{00} = \alpha_1 a_0 + \alpha_2 b_0 + \alpha_1\varphi_{00} + \alpha_2\phi_{00}$, $A_{10} = \alpha_1(a_1c_1 + a_2c_3) + \alpha_2(b_1c_1 + b_2c_3) + \alpha_1\varphi_{10} + \alpha_2\phi_{10}$, $A_{01} = \alpha_1(a_1c_2 + a_2c_4) + \alpha_2(b_1c_2 + b_2c_4) + \alpha_1\varphi_{01} + \alpha_2\phi_{01}$, $A_{n0} = \alpha_1\varphi_{n0} + \alpha_2\phi_{n0}$, $A_{0m} = \alpha_1\varphi_{0m} + \alpha_2\phi_{0m}$ and $A_{nm} = \alpha_1\varphi_{nm} + \alpha_2\phi_{nm}$, $\forall n, m \geqslant 1$.

Now, for the steady state, $z = z_s$, both $\dot{z}$ and $\ddot{z}$ vanish and the fixed points follow the condition

$$A_{00} + A_{10} z_s + \sum_{n>1}^{\infty} A_{n0} z_s^n = 0. \tag{3.7}$$

The equation for deviation from the stationary point from z i.e. $\xi(= z - z_s)$ follows from Eq. (3.6) as,

$$\ddot{\xi} + F(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0, \tag{3.8}$$

where the functions $F(\xi, \dot{\xi})$ and $G(\xi)$ are given by

$$\begin{aligned} F(\xi, \dot{\xi}) &= -\left[A_{01} + \sum_{m>1}^{\infty} A_{0m} \dot{\xi}^{m-1} + \sum_{n,m\geq 1}^{\infty} A_{nm}(\xi + z_s)^n \dot{\xi}^{m-1}\right] \\ G(\xi) &= -\left[A_{00} + A_{10}(\xi + z_s) + \sum_{n>1}^{\infty} A_{n0}(\xi + z_s)^n\right]. \end{aligned} \tag{3.9}$$

Eq. (3.8) is a well known form of generalised Liénard equation or Liénard–Levinson–Smith (LLS) system if the damping force, $F(\xi, \dot{\xi})$ and the restoring force, $G(\xi)$ satisfy the usual regularity conditions as given in Strogatz[10] page-210. So, the condition for existence of having a stable limit cycle of the above described Liénard system should satisfy $F(0, 0) < 0$ i.e.

$$-\left[A_{01} + \sum_{n\geqslant 1}^{\infty} A_{n1} z_s^n\right] < 0. \tag{3.10}$$

3.2.1 *Connection between Liénard–Levinson–Smith (LLS) system and stability theory*

It is well observed that a Liénard system or LLS system must contains a trivial fixed point as they are arbitrary second order homogeneous autonomous ordinary differential equation (ODE) and its oscillatory behaviour can be found near the origin. Now, let $\lambda_{1,2} = \alpha \pm i\beta$ be complex conjugate eigenvalues of stability matrix of (3.1), calculated near a fixed point, then the system near the fixed point will be asymptotically stable spiral or center or an unstable spiral if $\alpha < 0$, or $\alpha = 0$ or $\alpha > 0$, respectively. From the local bifurcation theory, it is known that, to have a stable limit cycle attractor in the phase space, the fixed point must have to be unstable in nature i.e. $\alpha > 0$. One can conclude the similar form of the sign of the coefficient of the non-zero damping i.e. $F(0, 0) < 0$ as there is a relation between the constant damping coefficient of a Liénard or LLS system with the real part of eigenvalues, is $F(0, 0) = -2\alpha$ i.e. the constant damping coefficient is directly proportional to the real part of the eigenvalue distinguishable by a sign. This has been able to provide successfully the same if any system can be written in a Liénard or LLS form. The solution will be a center in nature, if $\alpha = 0$ i.e. $F(0,0) = 0$ i.e. when there is no constant damping force in the system around the origin and for an asymptotically stable spiral solution, $F(0,0) > 0(\alpha < 0)$ implies that the solution starts near the limit cycle and ends spirally to the critical point. The direction of the spiral can be determined from the sign of the time derivative of the angular variable (in polar coordinate).

3.3 ISOCHRONICITY FOR LIÉNARD SYSTEM

For a given two-dimensional nonlinear dynamical system of equations, in general, can be cast into Liénard system. From the Liénard system one can set up a perturbation theory around the closed orbit of the center[48, 49, 58, 64]. The orbit is characterized by two constants, the amplitude, A and the phase, θ, fixed by the two initial conditions. However, the perturbation theory most likely diverges due to the presence of secular terms as the separation of time scale becomes large. Two renormalisation constants have to be introduced to absorb these divergence. The renormalisation constants appear in terms of an arbitrary time, say τ, which serves to fix the new initial condition which makes the amplitude and the phase then dependent on τ. The value of x at t cannot depend on where one sets the initial condition and hence $\left(\frac{\partial x}{\partial \tau}\right)_t = 0$, which is the flow equation. This must give $\frac{dA}{d\tau} = p(A)$ and $\frac{d\theta}{d\tau} = q(A)$. If the system is of center-type then the initial condition sets the amplitude of motion and hence $\frac{dA}{d\tau} = 0$. The phase flow equation on the other hand normally furnishes the nonlinear correction to the frequency. However, for an isochronous center there can be no correction to the frequency and hence $p(A) = 0$ and $q(A) = 0$ identically. This implies the amplitude, A is fixed once the initial condition is set[48]. Our objective is to derive the condition for isochronicity[48, 49] from a Liénard equation and finally the relation between the condition for being a stable limit cycle and the mutual relation.

First we find a simple Liénard system from (3.8) by taking some special order of the power series in which n, m contribute starting from 0 to atmost 2 and from that we may get a polynomial of highest degree atmost 3 in the damping force function for the Liénard system. Using the above assumption, one can obtain,

$$\begin{aligned} F(\xi,\dot{\xi}) &= -\left[A_{01} + \sum_{m>1}^{2} A_{0m}\dot{\xi}^{m-1} + \sum_{n,m\geq 1}^{2} A_{nm}(\xi+z_s)^n\dot{\xi}^{m-1}\right] \\ \Rightarrow \quad F(\xi,\dot{\xi}) &= -[A_{01} + A_{02}\dot{\xi} + A_{11}\xi + A_{11}z_s + A_{12}\xi\dot{\xi} + A_{12}z_s\dot{\xi} + A_{21}\xi^2 \\ &\quad +2A_{21}\xi z_s + A_{21}z_s^2 + A_{22}\xi^2\dot{\xi} + 2A_{22}\xi z_s\dot{\xi} + A_{22}z_s^2\dot{\xi}] \end{aligned} \tag{3.11}$$

and

$$G(\xi) = -\left[A_{00} + A_{10}\xi + A_{10}z_s + A_{20}\xi^2 + 2A_{20}\xi z_s + A_{20}z_s^2\right].$$

The condition for being stable limit cycle, $F(0,0) < 0$ gives,

$$-\left[A_{01} + A_{11}z_s + A_{21}z_s^2\right] < 0 \tag{3.12}$$

and (3.7) gives

$$A_{00} + A_{10}z_s + A_{20}z_s^2 = 0 \tag{3.13}$$

and therefore,

$$G(\xi) = -[A_{10}\xi + A_{20}\xi^2 + 2A_{20}\xi z_s].$$

As $G(\xi)$ must satisfy $G(\xi) = 0$ at $\xi = 0$, thus the Liénard system becomes,

$$\begin{aligned}\ddot{\xi} - [A_{01} + A_{02}\dot{\xi} + A_{11}\xi + A_{11}z_s + A_{12}\xi\dot{\xi} + A_{12}z_s\dot{\xi} + A_{21}\xi^2 + 2A_{21}\xi z_s + A_{21}z_s^2 \\ + A_{22}\xi^2\dot{\xi} + 2A_{22}\xi z_s\dot{\xi} + A_{22}z_s^2\dot{\xi}]\dot{\xi} - [A_{10}\xi + A_{20}\xi^2 + 2A_{20}\xi z_s] = 0.\end{aligned} \tag{3.14}$$

Now if we set the condition other than stable limit cycle i.e. $F(0,0) = 0$, for example, taking (3.12) as zero for some values of the parameters, then above equation on simplifying becomes,

$$\begin{aligned}\ddot{\xi} + \omega^2\xi &= A_{02}\dot{\xi}^2 + A_{11}\xi\dot{\xi} + A_{12}\xi\dot{\xi}^2 + A_{12}z_s\dot{\xi}^2 + A_{21}\xi^2\dot{\xi} + 2A_{21}\xi z_s\dot{\xi} \\ &\quad + A_{22}\xi^2\dot{\xi}^2 + 2A_{22}\xi z_s\dot{\xi}^2 + A_{22}z_s^2\dot{\xi}^2 + A_{20}\xi^2\end{aligned} \tag{3.15}$$

where $\omega^2 = -2A_{20}z_s - A_{10}$ must be $+ve$. Since ω is a real quantity and for $\omega^2 < 0$ then $G(\xi)$ violate its property.

For book keeping purpose we introduce a positive λ, with $\lambda << 1$ and using on (3.15) and discarding higher orders of λ we get,

$$\ddot{\xi} + \omega^2\xi = \lambda A_{20}\xi^2 + \lambda\left[A_{02} + A_{22}z_s^2 + A_{12}z_s\right]\dot{\xi}^2 + \lambda\left[A_{11} + 2A_{21}z_s\right]\xi\dot{\xi} + O(\lambda^2). \tag{3.16}$$

After that using RG technique[48], let us take a perturbation solution of $\xi = \xi_0 + \lambda\xi_1 + \lambda^2\xi_2 + \lambda^3\xi_3 + \cdots\cdots\cdots$, i.e., $\xi = \xi_0 + \lambda\xi_1 + O(\lambda^2)$, to get an approximate solution of (3.16). So, putting ξ and after simplifying (on neglecting $O(\lambda^2)$), we get,

$$\begin{aligned}(\ddot{\xi}_0 + \lambda\ddot{\xi}_1) + \omega^2(\xi_0 + \lambda\xi_1) = \lambda A_{20}\xi_0^2 + \lambda(A_{02} + A_{12}z_s + A_{22}z_s^2)\dot{\xi_0}^2 \\ + \lambda(A_{11} + 2A_{21}z_s)\xi_0\dot{\xi}_0 + O(\lambda^2).\end{aligned} \tag{3.17}$$

Comparing the coefficient of λ^0, λ^1 of both sides we get,

$$\begin{aligned} \lambda^0 : \ddot{\xi}_0 + \omega^2 \xi_0 &= 0 && (3.18) \\ \lambda^1 : \ddot{\xi}_1 + \omega^2 \xi_1 &= A_{20}\xi_0^2 + (A_{02} + A_{12}z_s + A_{22}z_s^2)\dot{\xi_0}^2 + (A_{11} + 2A_{21}z_s)\xi_0\dot{\xi}_0. && (3.19) \end{aligned}$$

If we take higher order terms then it must be included within $O(\lambda^2)$ and we simply neglect here $O(\lambda^2)$. Let us set an initial condition $\xi(t) = A$ and $\dot{\xi(t)} = 0$ at $t = t_0$ with t_0 being the initial time, then by comparing λ as previously we get $\xi_0 = A$ and $\xi_i = 0, \forall i > 0$ along with $\dot{\xi}_i = 0, \forall i \geq 0$ at $t = t_0$.

Thus after solving above equations we get $\xi_0(t)$ and $\xi_1(t)$ as,

$$\xi_0(t) = A\cos\omega(t - t_0)$$

$$\begin{aligned} \xi_1(t) = &- \left[\frac{A^2 A_{20}}{3\omega^2} + \frac{2A^2(A_{02} + A_{12}z_s + A_{22}z_s^2)}{3}\right]\cos\omega(t - t_0) \\ &- \frac{A^2(A_{11} + 2A_{21}z_s)\sin\omega(t - t_0)}{3\omega} + \frac{A^2 A_{20}}{2}\left[\frac{1}{\omega^2} - \frac{\cos 2\omega(t - t_0)}{3\omega^2}\right] \\ &+ \frac{(A_{02} + A_{12}z_s + A_{22}z_s^2)\omega^2 A^2}{2}\left[\frac{1}{\omega^2} + \frac{\cos 2\omega(t - t_0)}{3\omega^2}\right] \\ &+ \frac{A^2(A_{11} + 2A_{21}z_s)\sin 2\omega(t - t_0)}{6\omega}. \end{aligned} \quad (3.20)$$

So the approximate solution of $\xi(t)$ is,

$$\begin{aligned} \xi(t) = &A\cos\omega(t - t_0) - \lambda\left[\frac{A^2 A_{20}}{3\omega^2} + \frac{2A^2(A_{02} + A_{12}z_s + A_{22}z_s^2)}{3}\right]\cos\omega(t - t_0) \\ &- \lambda\frac{A^2(A_{11} + 2A_{21}z_s)\sin\omega(t - t_0)}{3\omega} + \lambda\frac{A^2 A_{20}}{2}\left[\frac{1}{\omega^2} - \frac{\cos 2\omega(t - t_0)}{3\omega^2}\right] \\ &+ \lambda\frac{(A_{02} + A_{12}z_s + A_{22}z_s^2)\omega^2 A^2}{2}\left[\frac{1}{\omega^2} + \frac{\cos 2\omega(t - t_0)}{3\omega^2}\right] \\ &+ \lambda\frac{A^2(A_{11} + 2A_{21}z_s)\sin 2\omega(t - t_0)}{6\omega} \end{aligned} \quad (3.21)$$

where A is the amplitude and ω is frequency supposing the constant $-\omega t_0 = \theta_0$. Then $\xi(t)$ becomes,

$$\begin{aligned} \xi(t) = &A\cos(\omega t + \theta_0) - \lambda\left[\frac{A^2 A_{20}}{3\omega^2} + \frac{2A^2(A_{02} + A_{12}z_s + A_{22}z_s^2)}{3}\right]\cos(\omega t + \theta_0) \\ &- \lambda\frac{A^2(A_{11} + 2A_{21}z_s)\sin(\omega t + \theta_0)}{3\omega} + \lambda\frac{A^2 A_{20}}{2}\left[\frac{1}{\omega^2} - \frac{\cos 2(\omega t + \theta_0)}{3\omega^2}\right] \\ &+ \lambda\frac{(A_{02} + A_{12}z_s + A_{22}z_s^2)\omega^2 A^2}{2}\left[\frac{1}{\omega^2} + \frac{\cos 2(\omega t + \theta_0)}{3\omega^2}\right] \\ &+ \lambda\frac{A^2(A_{11} + 2A_{21}z_s)\sin 2(\omega t + \theta_0)}{6\omega}. \end{aligned} \quad (3.22)$$

At this point add another perturbation in the time interval $(t - t_0)$, by splitting $(t - \tau) + (\tau - t_0)$, where $t_0 < \tau < t$ and τ is very close to t_0 by defining the interval $(t - \tau)$ as a principal part and the remaining part $(\tau - t_0)$ can be neglected because of smallness.

Suppose that taking perturbation the time interval, amplitude and phase will be slightly changed from A to $A(\tau)$ and θ_0 to $\theta(\tau)$. From RG technique the relation between them are $A(\tau) = \frac{A}{Z_1(\tau,t_0)}$ and $\theta(\tau) = \theta_0 - Z_2(\tau, t_0)$, where

$$Z_1(\tau, t_0) \quad = 1 + \textstyle\sum_1^\infty \lambda^n p_n \text{ and } Z_2(\tau, t_0) = 0 + \sum_1^\infty \lambda^n q_n. \tag{3.23}$$

Neglecting terms of $O(\lambda^2)$ we get,

$$Z_1(\tau, t_0) \quad = 1 + \lambda p_1 + O(\lambda^2) \text{ and } Z_2(\tau, t_0) = \lambda q_1 + O(\lambda^2). \tag{3.24}$$

Now if we put the function Z_1 and Z_2 as well as A and θ_0 in (3.22) and remove the terms which could led to divergence, we must get either p_1 is zero or anything containing $(\tau - t_0)$ and the same for q_1 also. But because of the smallness of $(\tau - t_0)$, we can take p_1 and q_1 approximately to be zero. So, after considering above, the constants A become $A(\tau)$ and θ_0 become $\theta(\tau)$, i.e. they become dependent upon the time variable, τ. Also, if any term multiplied directly by $(t - t_0)$ in the final solution of $\xi(t)$, then we can convert it into $(t - \tau)$ by neglecting the other part. But here no such terms are directly involved in this solution. So $\xi(t)$ becomes,

$$\begin{aligned}
\xi(t) = {} & A(\tau)\cos(\omega t + \theta(\tau)) \\
& - \lambda \left[\frac{A^2(\tau) A_{20}}{3\omega^2} + \frac{2A^2(\tau)(A_{02} + A_{12}z_s + A_{22}z_s^2)}{3} \right] \cos(\omega t + \theta(\tau)) \\
& - \lambda \frac{A^2(\tau)(A_{11} + 2A_{21}z_s)\sin(\omega t + \theta(\tau))}{3\omega} + \lambda \frac{A^2(\tau)A_{20}}{2} \left[\frac{1}{\omega^2} - \frac{\cos 2(\omega t + \theta(\tau))}{3\omega^2} \right] \\
& + \lambda \frac{(A_{02} + A_{12}z_s + A_{22}z_s^2)\omega^2 A^2(\tau)}{2} \left[\frac{1}{\omega^2} + \frac{\cos 2(\omega t + \theta(\tau))}{3\omega^2} \right] \\
& + \lambda \frac{A^2(\tau)(A_{11} + 2A_{21}z_s)\sin 2(\omega t + \theta(\tau))}{6\omega}.
\end{aligned} \tag{3.25}$$

Since the final solution cannot depend on the arbitrary time scale, τ, we impose the condition $(\frac{\partial \xi}{\partial \tau})_t = 0$ which leads to

$$\frac{dA}{d\tau} \quad = 0 \text{ and } \tfrac{d\theta}{d\tau} = 0. \tag{3.26}$$

The independence of θ upon τ i.e. $\frac{d\theta}{d\tau} = g(A) = 0$ gives the condition for isochronicity. If it is non-zero then the Liénard system would not be isochronous. Thus the system will be isochronous for any values of A only when $F(0,0) = 0$.

Further, if $\frac{dA}{d\tau} = f(A)$ then we can say there be a limit cycle if $f(A) \neq 0$ and the radius of the cycle can be obtained by making $f(A) = 0$ if any non-zero A is found. Otherwise we cannot have any limit cycle because we cannot get any idea about the radii of the cycle. If this type of difficulty comes then we may call this as a center-type. When $\frac{dA}{d\tau} = f(A) = 0$ then it is also called center-type.

So, it is seen that, by pushing the condition $F(0,0) = 0$ which is the constant portion present in the damping force, the Liénard-type limit cycle oscillator transforms into an isochronous oscillator and this is the only condition for being isochronous oscillator. For $F(0,0) = 0$, finally it shows that the Liénard system looses its stability as limit cycle and becomes a center-type.

3.4 SOME CHEMICAL OSCILLATOR MODELS

Here we consider a few chemical oscillator models as examples of open system. In open systems there are some inputs and outputs, however, it is possible to attain a steady state depending upon the values of the parameters in addition to dynamical complexities due to the nonlinearities of the system of equations. Inspired by the above analysis of Liénard system we now study some examples to check and verify the above results of limit cycle and isochronicity.

3.4.1 *Modified Brusselator model*

The classical Brusselator model[33, 36, 50] is known to exhibit kinetics of model tri-molecular irreversible reactions which are based on the vast studies of chemical oscillations[18–21, 75, 76] in various systems. The reduction of the Brusselator model in the form of Rayleigh[15, 22] and Liénard form[40] of differential equations are already published. Here we have shown the condition of limit cycle and isochronicity for a modified Brusselator model.

The original four variable reversible Brusselator model[50] which after appropriate elimination of variable results in a simple kinetics of relevant two variables[15] as

$$\begin{aligned} \frac{dx}{dt} &= a_1 + x^2 y - (\alpha + b)x, \\ \frac{dy}{dt} &= bx - x^2 y, \end{aligned} \tag{3.27}$$

where x and y are the dimensionless concentration of some species. The parameters $a_1, b, \alpha > 0$ follow the properties: $a_1 = \mu(1-\beta)a + \mu\beta$ with $a_1 > 0 \Rightarrow$ either $\mu > 0$ and $a < \frac{\beta}{1-\beta}$ or $\mu < 0$ and $a > \frac{\beta}{1-\beta}$. Depending on the values of β one can choose the conditions accordingly.

Supposing $z = x + y$ and $u = a_1 - \alpha x$ one can transform (3.27) into one equation i.e.

$$\dot{z} = u. \tag{3.28}$$

Here x and y can be expressed as

$$x \quad = \tfrac{a_1 - u}{\alpha} \text{ and } y = z + \tfrac{u - a_1}{\alpha}. \tag{3.29}$$

Differentiating (3.28) w.r.t. t we get,

$$\begin{aligned} \ddot{z} &= \dot{u} = -\alpha a_1 - \alpha x^2 y + \alpha(\alpha + b)x \\ \Rightarrow \ddot{z} &= \left(ba_1 + \frac{a_1^3}{\alpha^2}\right) - \frac{za_1^2}{\alpha} + \frac{2a_1 z\dot{z}}{\alpha} - \dot{z}\left(\alpha + b + \frac{3a_1^2}{\alpha^2}\right) + \frac{3a_1\dot{z}^2}{\alpha^2} - \frac{\dot{z}^3}{\alpha^2} - \frac{z\dot{z}^2}{\alpha}. \end{aligned} \tag{3.30}$$

So if z_s be the stationary point then $\dot{z}$ and $\ddot{z}$ all are zero which shows from (3.30) as,

$$z_s = \frac{\alpha}{a_1^2}\left(ba_1 + \frac{a_1^3}{\alpha^2}\right). \tag{3.31}$$

Taking perturbation around the fixed point z_s i.e. $z = z_s + \xi$ with $\dot{z} = \dot{\xi}$ and $\ddot{z} = \ddot{\xi}$ and substituting this in (3.30) and on simplification gives

$$\ddot{\xi} + F(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0 \tag{3.32}$$

where the functions, $F(\xi, \dot{\xi})$ and $G(\xi)$ are

$$\begin{aligned} F(\xi, \dot{\xi}) &= -\frac{2a_1\xi}{\alpha} - b + \frac{a_1^2}{\alpha^2} + \alpha - \frac{2a_1\dot{\xi}}{\alpha^2} + \frac{b\dot{\xi}}{a_1} + \frac{\dot{\xi}^2}{\alpha^2} + \frac{\xi\dot{\xi}}{\alpha}; \\ G(\xi) &= \frac{a_1^2\xi}{\alpha}. \end{aligned} \tag{3.33}$$

This is Liénard system because all the conditions for being a Linéard oscillator which are stated previously are satisfied by using the given conditions $a_1, b, \alpha > 0$. Thus if there is any stable limit cycle then it must satisfy $F(0,0) < 0$ i.e.

$$a_1^2 < (b - \alpha)\alpha^2. \tag{3.34}$$

Now (3.32) can be written as,

$$\ddot{\xi} + \omega^2\xi = \frac{2a_1\xi\dot{\xi}}{\alpha} + \left(b - \frac{a_1^2}{\alpha^2} - \alpha\right)\dot{\xi} + \left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1}\right)\dot{\xi}^2 - \frac{\dot{\xi}^3}{\alpha^2} - \frac{\xi\dot{\xi}^2}{\alpha}; \ \omega^2 = \frac{a_1^2}{\alpha} > 0. \tag{3.35}$$

For book keeping purpose introducing $\lambda(0 < \lambda << 1)$ in such a way that the above equation can be expressed as,

$$\ddot{\xi} + \omega^2 \xi = \lambda \frac{2a_1 \xi \dot{\xi}}{\alpha} + \left(b - \frac{a_1^2}{\alpha^2} - \alpha \right) \dot{\xi} + \lambda \left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1} \right) \dot{\xi}^2 + O(\lambda^2) \tag{3.36}$$

(neglecting $O(\lambda^2)$ included terms)

To solve the equation by applying RG technique, we can take $\xi = \xi_0 + \lambda \xi_1 + \lambda^2 \xi_2 + \cdots$ i.e. $\xi = \xi_0 + \lambda \xi_1 + O(\lambda^2)$, by neglecting $O(\lambda^2)$ terms. So putting ξ in (3.36) and equating the coefficients of both sides for λ^0 and λ^1 we get,

$$\begin{aligned}
\lambda^0 : \ddot{\xi}_0 + \omega^2 \xi_0 &= \left(b - \frac{a_1^2}{\alpha^2} - \alpha \right) \dot{\xi}_0, \\
\lambda^1 : \ddot{\xi}_1 + \omega^2 \xi_1 &= \frac{2a_1 \xi_0 \dot{\xi}_0}{\alpha} + \left(b - \frac{a_1^2}{\alpha^2} - \alpha \right) \dot{\xi}_1 + \left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1} \right) \dot{\xi_0}^2.
\end{aligned} \tag{3.37}$$

Now if we treat the constant coefficient in $F(\xi, \dot{\xi})$ as zero i.e. $F(0,0) = 0$ or, $\left(b - \frac{a_1^2}{\alpha^2} - \alpha \right) = 0$ then from (3.37) we have

$$\begin{aligned}
\lambda^0 : \ddot{\xi}_0 + \omega^2 \xi_0 &= 0, \\
\lambda^1 : \ddot{\xi}_1 + \omega^2 \xi_1 &= \frac{2a_1 \xi_0 \dot{\xi}_0}{\alpha} + \left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1} \right) \dot{\xi_0}^2.
\end{aligned} \tag{3.38}$$

Setting initial condition $\xi(t) = A$ and $\xi(\dot{t}) = 0$ at $t = t_0$, then by comparing λ we get $\xi_0 = A$ and $\xi_i = 0, \forall i > 0$ and $\dot{\xi}_i = 0, \forall i \geq 0$ at $t = t_0$. After solving (3.38), $\xi_0(t)$ and $\xi_1(t)$ becomes,

$$\xi_0(t) = A \cos \omega(t - t_0),$$

$$\begin{aligned}
\xi_1(t) = {} & \frac{A^2}{2} \left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1} \right) \{1 + \frac{\cos 2\omega(t - t_0)}{3} - \frac{4 \cos \omega(t - t_0)}{3}\} \\
& + \frac{A^2 a_1}{3\omega\alpha} \{\sin 2\omega(t - t_0) - 2 \sin \omega(t - t_0)\}.
\end{aligned} \tag{3.39}$$

So, $\xi(t)$ becomes (on using $\theta_0 = -\omega t_0$),

$$\begin{aligned}
\xi(t) = A \cos(\omega t + \theta_0) + \lambda \Bigg[& \frac{A^2}{2} \left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1} \right) \{1 + \frac{\cos 2(\omega t + \theta_0)}{3} - \frac{4 \cos(\omega t + \theta_0)}{3}\} \\
& + \frac{A^2 a_1}{3\omega\alpha} \{\sin 2(\omega t + \theta_0) - 2 \sin(\omega t + \theta_0)\} \Bigg].
\end{aligned} \tag{3.40}$$

Now considering perturbation in the time interval $(t - t_0)$ by splitting $(t - \tau) + (\tau - t_0)$, where $t_0 < \tau < t$ and τ is very close to t_0, we define the interval $(t - \tau)$ as principal part and the remaining small part, $(\tau - t_0)$ can be neglected.

Suppose here considering perturbation of the time interval the amplitude and the phase slightly change from A to $A(\tau)$ and θ_0 to $\theta(\tau)$. From RG technique the relation between them are $A(\tau) = \frac{A}{Z_1(\tau, t_0)}$ and $\theta(\tau) = \theta_0 - Z_2(\tau, t_0)$ where

$$Z_1(\tau, t_0) \quad = 1 + \sum_1^\infty \lambda^n p_n \text{ and } Z_2(\tau, t_0) = 0 + \sum_1^\infty \lambda^n q_n. \tag{3.41}$$

Since we are neglecting $O(\lambda^2)$ then from previous equation we get,

$$Z_1(\tau, t_0) \quad = 1 + \lambda p_1 + O(\lambda^2) \text{ and } Z_2(\tau, t_0) = \lambda q_1 + O(\lambda^2). \tag{3.42}$$

If we put the functions Z_1 and Z_2 as well as A and θ_0 in (3.40) and remove the terms which could lead to divergence we must get either p_1 is zero or anything containing $(\tau - t_0)$ and finally for q_1. But because of the smallness of $(\tau - t_0)$ we can take p_1 and q_1 approximately zero. So, after using the above argument the constant, A becomes $A(\tau)$ and constant θ_0 becomes $\theta(\tau)$ i.e. they become dependent upon the time variable τ. Also, if any term multiplied directly by $(t - t_0)$ in the final solution of $\xi(t)$, then we can convert it into $(t - \tau)$ by neglecting the non-principal part.

Using all above considerations in Eq. (3.40), $\xi(t)$ becomes,

$$\begin{aligned}\xi(t) = {} & A(\tau)\cos(\omega t + \theta(\tau)) \\ & + \lambda\left[\frac{A^2(\tau)}{2}\left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1}\right)\{1 + \frac{\cos 2(\omega t + \theta(\tau))}{3} - \frac{4\cos(\omega t + \theta(\tau))}{3}\}\right. \\ & \left. + \frac{A^2(\tau)a_1}{3\omega\alpha}\{\sin 2(\omega t + \theta(\tau)) - 2\sin(\omega t + \theta(\tau))\}\right].\end{aligned} \tag{3.43}$$

So, finally under the condition $\left(\frac{\partial\xi}{\partial\tau}\right)|_t = 0$, since the final solution can not be dependent on τ, (3.43) shows,

$$\begin{aligned} & \left[\cos(\omega t + \theta(\tau))\right. \\ & + \lambda A(\tau)\left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1}\right)\{1 + \frac{\cos 2(\omega t + \theta(\tau))}{3} - \frac{4\cos(\omega t + \theta(\tau))}{3}\} \\ & \left. + \frac{2\lambda A(\tau)a_1}{3\omega\alpha}\{\sin 2(\omega t + \theta(\tau)) - 2\sin(\omega t + \theta(\tau))\}\right]\frac{dA}{d\tau} \\ & + \left[-A(\tau)\sin(\omega t + \theta(\tau)) + \frac{\lambda A^2(\tau)}{2}\left(\frac{2a_1}{\alpha^2} - \frac{b}{a_1}\right)\{-\frac{2\sin 2((\omega t + \theta(\tau))}{3}\right. \\ & \left. + \frac{4\sin(\omega t + \theta(\tau))}{3}\} + \frac{2\lambda A^2(\tau)a_1}{3\omega\alpha}\{\cos 2(\omega t + \theta(\tau)) - \cos(\omega t + \theta(\tau))\}\right]\frac{d\theta}{d\tau} = 0.\end{aligned} \tag{3.44}$$

Since, $A(\tau) \neq 0, a_1 \neq 0, \alpha \neq 0, b \neq 0, \omega \neq 0$, then none of the above which are in third brackets are zero. Therefore the only possible way to balance the equation is $\frac{dA}{d\tau} = 0$ and $\frac{d\theta}{d\tau} = 0$. So, this leads to isochronous oscillator of center-type and finally it cannot have any limit cycle. Since, we know that $a_1, b, \alpha > 0$ and $a_1 = \mu(1-\beta)a + \mu\beta$ so this will give either $\mu > 0$ and $a < \frac{\beta}{1-\beta}$ or $\mu < 0$ and $a > \frac{\beta}{1-\beta}$. The parameters a and β are dependent on each other. Now we deal with the positive region of parameters and we suppose $\mu = 1$ and $a = 1$ which gives $\beta > 0.5$. For this set of parametric values the boundary condition satisfies, $a_1^2 = (b-\alpha)\alpha^2$ which produces Fig. 3.1 in where parametric variation of alpha(α) and b are shown in which the boundary line separates the region into stable limit cycle and stable focus centerd for the modified Brusselator model. Fig. 3.2 shows a stable limit cycle solution for suitable choice of parameters, $\mu = 1$, $a = 1$, $\beta = 0.6$, $\alpha = 2$ and $b = 2.5$ together which satisfies the limit cycle condition, $F(0,0) < 0$. Fig. 3.3 shows a center-type solution satisfying $F(0,0) = 0$ by taking suitable choice of parameters, $\mu = 1$, $a = 1$, $\beta = 0.6$, $\alpha = 2$ and $b = 2.25$. Since Fig. 3.3 is a center-type, it must be closer to the fixed point but not form any limit cycle.

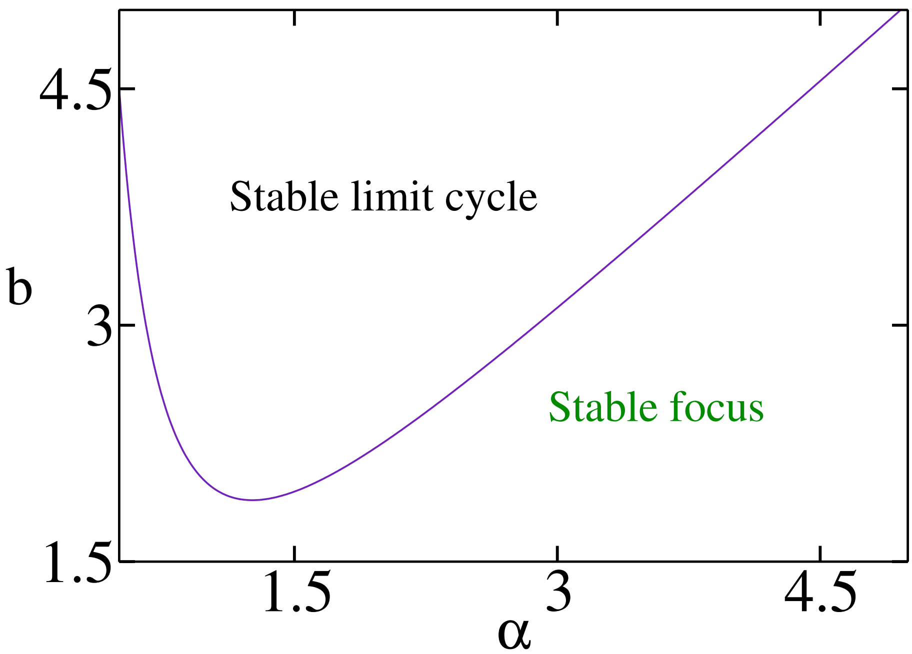

Figure 3.1: *Modified Brusselator model:* Parametric space diagram for α and b in which the boundary line separates the region into stable limit cycle and stable focus when $\mu = 1$, $a = 1$ and $\beta = 0.6$.

3.4.2 *Glycolytic oscillator*

The Glycolytic oscillator[6, 27, 77–79] is mainly observed in the yeast, which is described with respect to its overall dynamics and biochemical properties of its enzyme phospho fruc-

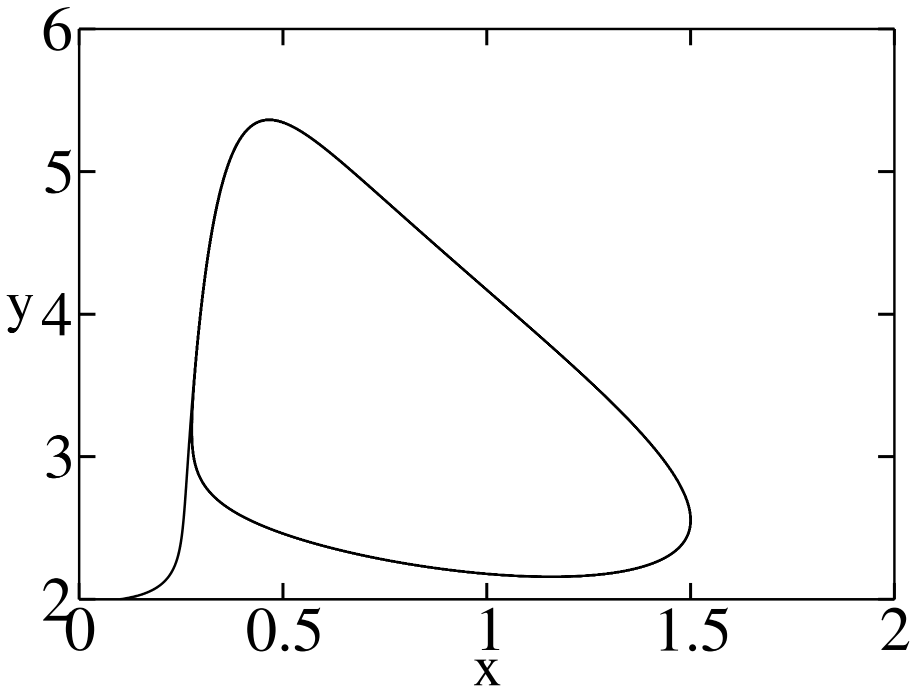

Figure 3.2: *Modified Brusselator model:* Phase portrait of (3.27) gives a stable limit cycle for suitable choice of parameters, $\mu = 1$, $a = 1$, $\beta = 0.6$, $\alpha = 2$, $b = 2.5$ together which satisfies the limit cycle condition, $F(0,0) < 0$.

tokinase. Kinetic properties are complemented by the mathematical analysis of Sel'kov[6, 27] and related models[77]. Here we have considered its modified form for the oscillatory Glycolysis in closed vessels by Merkin-Needham-Scott (MNS)[77] as

$$\begin{aligned} \dot{x} &= -x + (a + x^2)y \\ \dot{y} &= b - (a + x^2)y. \end{aligned} \tag{3.45}$$

with x and y corresponding to the intermediate species concentrations. The phosphofructokinase step considered by Sel'kov's model and its MNS-generalization considers ATP to ADP transition accompanying fructose-6-phosphate(F6P) to fructose-1,6-diphosphate(F1,6DP). The parameter b means ATP influx and a is the rate of non-catalyzed side-steps (a side-process, which needs to be taken into account for the closed vessel consideration, as shown by MNS[77]). The fixed point of the system is at $x = b, y = \frac{b}{a+b^2}$. It is stable focus for a certain parameter range and an unstable focus for certain others. The crossover from stable to unstable focus occurs on the boundary curve which is a locus of points in the $a - b$ plane where a Hopf bifurcation occurs i.e. the fixed point for those values of (a, b) is a center which satisfies the equation $(a + b^2)^2 + (a - b^2) = 0$ and can be obtained by checking condition for stability or from the eigenvalues.

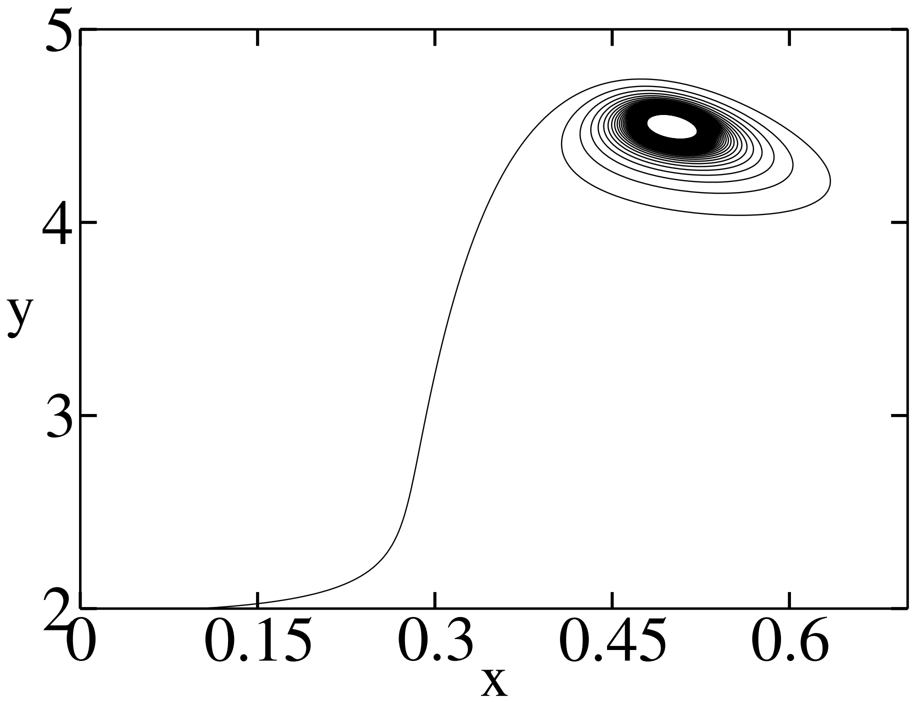

Figure 3.3: *Modified Brusselator model:* Phase portrait of (3.27) gives a center-type solution when $F(0,0) = 0$ by taking suitable choice of parameters, $\mu = 1$, $a = 1$, $\beta = 0.6$, $\alpha = 2$, $b = 2.25$.

If we suppose $z = x + y$ and $u = b - x$ then we can transform (3.45) into a form i.e.

$$\dot{z} = u. \tag{3.46}$$

Here x, y can be expressed as

$$x = (b - u) \text{ and } y = (z - b + u). \tag{3.47}$$

Differentiating (3.46) w.r.t. t and eliminating x and y gives,

$$\ddot{z} = -(1 + a + 3b^2)\dot{z} - (a + b^2)z + (b + ab + b^3) + 2bz\dot{z} + 3b\dot{z}^2 - z\dot{z}^2 - \dot{z^3} \tag{3.48}$$

If z_s be the fixed point of z, for which $\dot{z}, \ddot{z}$ all are zero then,

$$z_s = b + \frac{b}{a + b^2} = k(\text{say}). \tag{3.49}$$

Similarly as in previous case we take a perturbation around k i.e. $z = k + \xi$, $\dot{z} = \dot{\xi}$, $\ddot{z} = \ddot{\xi}$, then (3.48) gives a Liénard system,

$$\ddot{\xi} + F(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0 \tag{3.50}$$

where

$$\begin{aligned} F(\xi,\dot{\xi}) &= (1+a+3b^2) - 2b\xi - 2bk - 3b\dot{\xi} + \xi\dot{\xi} + k\dot{\xi} + \dot{\xi}^2 \quad \text{and} \\ G(\xi) &= (a+b^2)\xi. \end{aligned} \tag{3.51}$$

Note that all the conditions for a Liénard system are satisfied for suitable choice of a and b which is also obvious for $a \geq 0$ whatever b may be. So the condition for existence of stable limit cycle is $F(0,0) < 0$ i.e.

$$(a+b^2) + \frac{(a-b^2)}{(a+b^2)} < 0. \tag{3.52}$$

Suppose constant coefficient present in $F(\xi,\dot{\xi})$ is taken as zero then (3.50) shows,

$$\begin{aligned} &\ddot{\xi} - \{2b\xi + 3b\dot{\xi} - \xi\dot{\xi} - k\dot{\xi} - \dot{\xi}^2\}\dot{\xi} + (a+b^2)\xi = 0 \\ &\Rightarrow \ddot{\xi} + (a+b^2)\xi = 2b\xi\dot{\xi} + 3b\dot{\xi}^2 - \xi\dot{\xi}^2 - k\dot{\xi}^2 - \dot{\xi}^3, \end{aligned} \tag{3.53}$$

Similarly we consider

$$\ddot{\xi} + (a+b^2)\xi = 2\lambda b\xi\dot{\xi} + 3\lambda b\dot{\xi}^2 - \lambda^2\xi\dot{\xi}^2 - k\lambda\dot{\xi}^2 - \lambda^2\dot{\xi}^3 \tag{3.54}$$

where $\lambda(0 < \lambda << 1)$. So, because of smallness of λ, neglecting $O(\lambda^2)$, one finds

$$\ddot{\xi} + (a+b^2)\xi = 2\lambda b\xi\dot{\xi} + 3\lambda b\dot{\xi}^2 - k\lambda\dot{\xi}^2 + O(\lambda^2). \tag{3.55}$$

Taking a perturbative solution of ξ as $\xi = \xi_0 + \lambda\xi_1 + \lambda^2\xi_2 + \lambda^3\xi_3 + \cdots$ i.e. $\xi = \xi_0 + \lambda\xi_1 + O(\lambda^2)$ (neglecting $O(\lambda^2)$) and using above perturbative solution and comparing the coefficients of λ^0, λ^1 (3.55) gives,

$$\begin{aligned} \lambda^0 : \ddot{\xi}_0 + (a+b^2)\xi_0 &= 0 \\ \lambda^1 : \ddot{\xi}_1 + (a+b^2)\xi_1 &= 2b\xi_0\dot{\xi}_0 + (3b-k)\dot{\xi_0}^2. \end{aligned} \tag{3.56}$$

Using the most general initial condition $\xi(t) = A$ and $\dot{\xi(t)} = 0$ at $t = t_0$, then by comparing λ as similar as in previous case we must get $\xi_0 = A$ and $\xi_i = 0, \forall i > 0$ with $\dot{\xi}_i = 0, \forall i \geq 0$ at $t = t_0$.

After solving (3.56), $\xi_0(t)$ and $\xi_1(t)$ becomes,

$$\xi_0(t) = A\cos\omega(t-t_0)$$

and

$$\xi_1(t) = -\frac{2bA^2}{3\omega}\sin\omega(t-t_0) - \frac{2(3b-k)A^2}{3}\cos\omega(t-t_0)$$
$$+\frac{(3b-k)A^2}{2}\{1+\frac{\cos 2\omega(t-t_0)}{3}\} + \frac{bA^2}{3\omega}\sin 2\omega(t-t_0) \tag{3.57}$$

where $\omega^2 = (a+b^2) > 0$. So $\xi(t)$ becomes,

$$\xi(t) = A\cos\omega(t-t_0) + \lambda\left[-\frac{2bA^2}{3\omega}\sin\omega(t-t_0) - \frac{2(3b-k)A^2}{3}\cos\omega(t-t_0)\right.$$
$$\left.+\frac{(3b-k)A^2}{2}\{1+\frac{\cos 2\omega(t-t_0)}{3}\} + \frac{bA^2}{3\omega}\sin 2\omega(t-t_0)\right]. \tag{3.58}$$

If $\theta_0 = -\omega t_0$, then the above equation can be written as,

$$\xi(t) = A\cos(\omega t+\theta_0) + \lambda\left[-\frac{2bA^2}{3\omega}\sin(\omega t+\theta_0) - \frac{2(3b-k)A^2}{3}\cos(\omega t+\theta_0)\right.$$
$$\left.+\frac{(3b-k)A^2}{2}\{1+\frac{\cos 2(\omega t+\theta_0)}{3}\} + \frac{bA^2}{3\omega}\sin 2(\omega t+\theta_0)\right]. \tag{3.59}$$

Now adding another perturbation in the time interval $(t-t_0)$ by splitting $(t-\tau)+(\tau-t_0)$, where $t_0 < \tau < t$ and τ is very close to t_0 we define the interval $(t-\tau)$ as a principal part and the remaining part $(\tau-t_0)$ can be neglected because of the smallness. Suppose that on taking perturbation the time interval the amplitude and the phase slightly be changed from A to $A(\tau)$ and θ_0 to $\theta(\tau)$. From RG technique the relation between them are $A(\tau) = \frac{A}{Z_1(\tau,t_0)}$, and $\theta(\tau) = \theta_0 - Z_2(\tau, t_0)$; where

$$Z_1(\tau,t_0) \quad = 1+\textstyle\sum_1^\infty \lambda^n p_n \text{ and } Z_2(\tau,t_0) = 0+\sum_1^\infty \lambda^n q_n. \tag{3.60}$$

Since we are neglecting $O(\lambda^2)$ then from previous equation we get,

$$Z_1(\tau,t_0) \quad = 1+\lambda p_1 + O(\lambda^2) \text{ and } Z_2(\tau,t_0) = \lambda q_1 + O(\lambda^2). \tag{3.61}$$

Now, if we put the functions Z_1 and Z_2 as well as A and θ_0 in (3.59) and remove the terms which could lead to divergence we must get either p_1 is zero or anything containing $(\tau-t_0)$ and same for q_1. But because of the smallness of $(\tau-t_0)$ we can take p_1 and q_1 approximately to zero.Using all above results in Eq. (3.59) we get,

$$\xi(t) = A(\tau)\cos(\omega t+\theta(\tau)) + \lambda\left[-\frac{2bA^2(\tau)}{3\omega}\sin(\omega t+\theta(\tau)) - \frac{2(3b-k)A^2(\tau)}{3}\cos(\omega t+\theta(\tau))\right.$$
$$\left.+\frac{(3b-k)A^2(\tau)}{2}\{1+\frac{\cos 2(\omega t+\theta(\tau))}{3}\} + \frac{bA^2(\tau)}{3\omega}\sin 2(\omega t+\theta(\tau))\right] \tag{3.62}$$

So finally under the condition as in RG method $\left(\frac{\partial \xi}{\partial \tau}\right)|_t = 0$ (3.62) gives ,

$$\begin{aligned}\frac{dA}{d\tau} &= 0 \text{ and}\\ \frac{d\theta}{d\tau} &= 0.\end{aligned}$$

Since, $a \neq 0, b \neq 0 (\Rightarrow \omega \neq 0)$ and $A(\tau) \neq 0$ then none of the above in brackets in the last equation are zero which are obtained from RG condition. Therefore the only possible way is balancing the equation by making them zero. So this leads to isochronous center-type and finally it cannot have any limit cycle. So we can construct an isochronous oscillatory equation from Liénard system by suitable choice of parameters which makes $F(0,0) = 0$.

Fig. 3.5 gives a stable limit cycle for describing Glycolytic oscillator model when a and b are chosen as 0.11 and 0.6, respectively, together satisfies limit cycle condition. Fig. 3.6 represents a center-type solution when a is fixed at zero and and b is fixed at 1 (which are on the boundary point of Fig. 3.4 together which satisfies $F(0,0) = 0$).

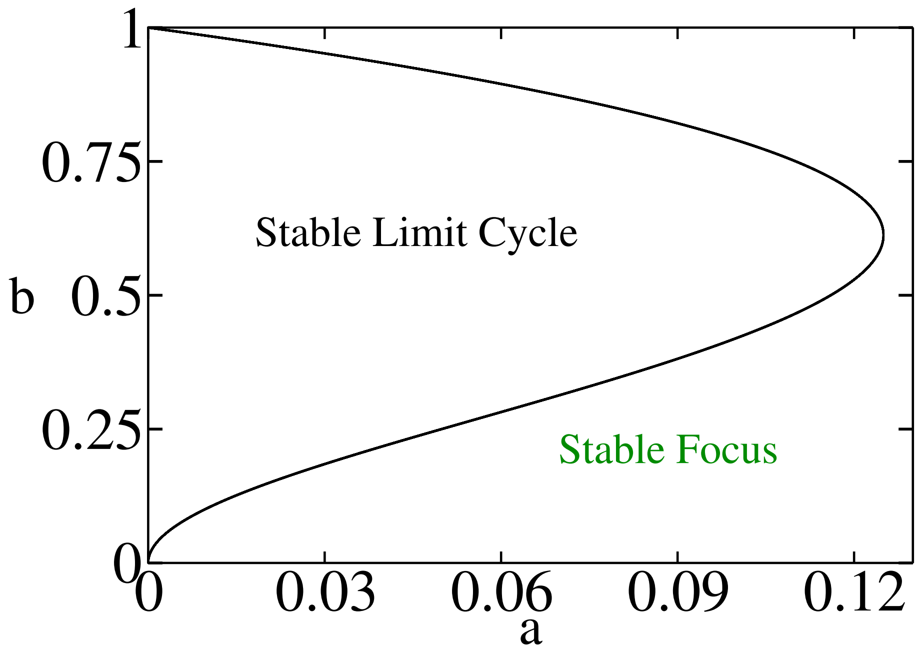

Figure 3.4: *Glycolytic oscillator:* Parametric phase portrait for the parameters, a and b in which the boundary line separates the region into stable limit cycle(inner side) and stable focus(outer side).

3.4.3 *Van der Pol type oscillator model*

Here we take an example of Van der Pol type oscillator. Van der Pol oscillator[10, 12, 46, 49] arises in many nonlinear dynamical systems including in chemical oscillation with cubic

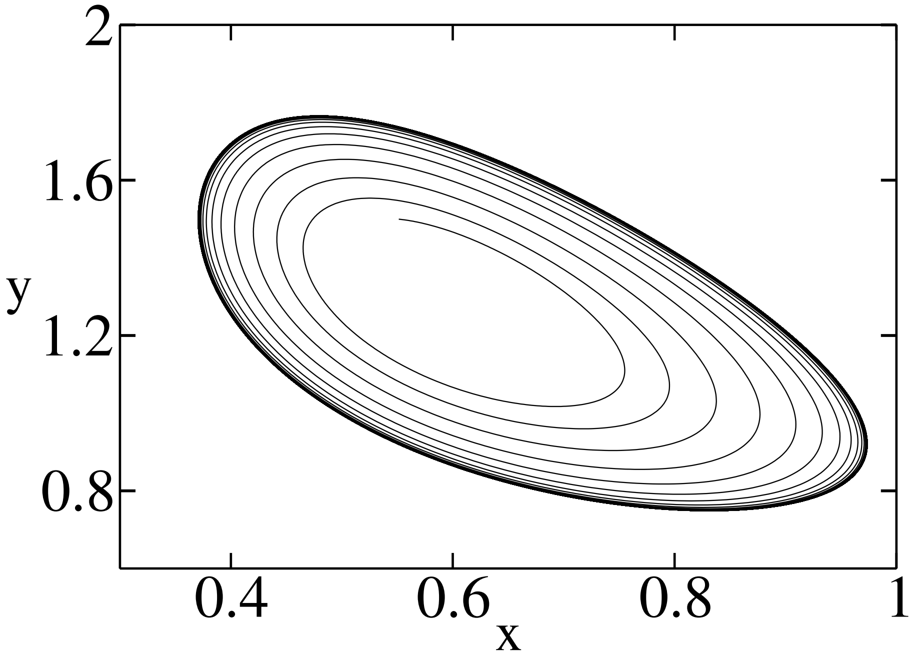

Figure 3.5: *Glycolytic oscillator:* Phase space diagram of (3.45) when a=0.11 and b=0.6 satisfies limit cycle condition, $F(0,0) < 0$ and gives a stable limit cycle.

nonlinear processes. This case is readily convertable to a Liénard oscillator form. Here we show the conditions of limit cycle and isochronicity with a slightly different analysis than the previous examples. The set of differential equation for Van der Pol type oscillator is given as,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\epsilon y(x^2 - a^2) - \omega^2 x, a \in \mathbb{R}. \end{aligned} \tag{3.63}$$

Originally in the Van der Pol oscillator equation a is 1. The size of the limit cycle depends on the magnitude of a. Here our analysis is valid for $a \in \mathbb{R}$, however, for the purpose of RG analysis[46, 47] here we consider a little generalized form with ϵ as a smallness parameter of perturbation although fixed by the Van der Pol system. There is a fixed point at the origin which has a stable focus for $\epsilon < 0$ and unstable focus for $\epsilon > 0$. The fixed point is a center for $\epsilon = 0$. If we differentiate first equation w.r.t. t and use second equation we get

$$\ddot{x} + \epsilon\dot{x}(x^2 - a^2) + \omega^2 x = 0. \tag{3.64}$$

It is of Liénard form and gives a Van der Pol oscillator for $a = 1$. We now analyze the model without taking any particular value of a. For this model the damping force is $F(x, \dot{x}) = \epsilon(x^2 - a^2)$ and $G(x) = \omega^2 x$. So they satisfy all the conditions for being a Liénard system which can be a limit cycle if $F(0,0) < 0$ i.e. $\forall a \in \mathbb{R}_{\neq 0}$ and $\epsilon > 0$ (ϵ is very small). If $\epsilon < 0$ then

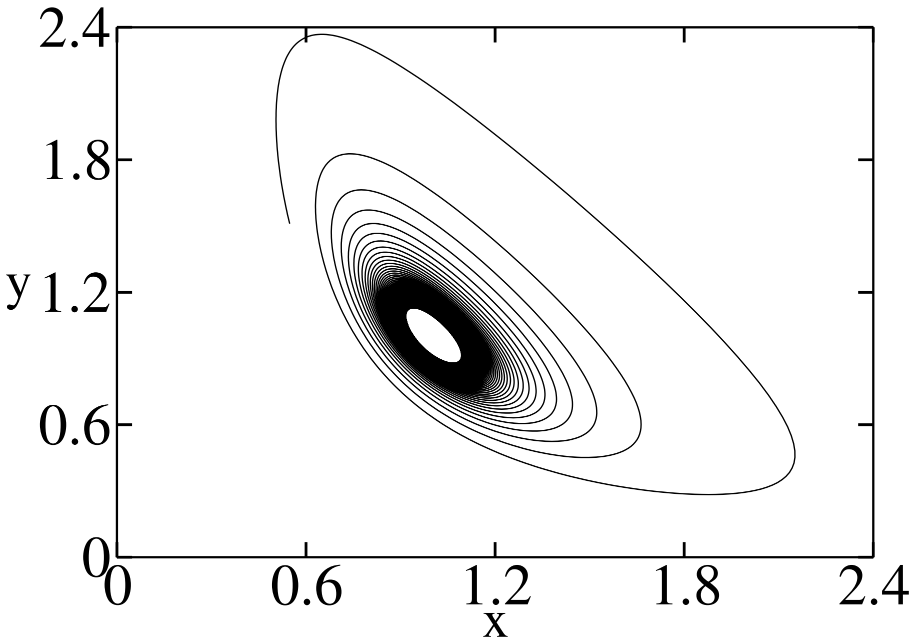

Figure 3.6: *Glycolytic oscillator:* Phase space diagram of (3.45) when $a = 0$ and $b = 1$ together which satisfies $F(0,0) = 0$ and gives a center i.e. a, b are on the boundary line of Fig. 3.4.

$a^2 < 0$, which can not be possible for any real a. Now for further calculation we rewrite the Eq. (3.64) as,

$$\ddot{x} + \omega^2 x = -\epsilon \dot{x}(x^2 - a^2). \tag{3.65}$$

Expanding $x(t)$ as $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$, i.e. $x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2)$, then neglecting $O(\epsilon^2)$ $(0 < \epsilon << 1)$ and from (3.65) by comparing the coefficients of ϵ^0, ϵ^1 we get,

$$\begin{aligned} \epsilon^0 : \ddot{x}_0 + \omega^2 x_0 &= 0 \\ \epsilon^1 : \ddot{x}_1 + \omega^2 x_1 &= -\dot{x}_0(x_0^2 - a^2). \end{aligned} \tag{3.66}$$

Using the general initial condition, $x(t) = A$ and $\dot{x(t)} = 0$ at $t = t_0$, then by comparing ϵ as similar in previous case we get $x_0 = A$ and $x_i = 0, \forall i > 0$ along with $\dot{x}_i = 0, \forall i \geq 0$ at $t = t_0$. After solving (3.66), $x_0(t)$ and $x_1(t)$ becomes,

$$x_0(t) = A \cos \omega(t - t_0)$$

and

$$x_1(t) = \tfrac{(7A^3 - 16Aa^2)}{32\omega} \sin \omega(t - t_0) - \tfrac{(A^3 - 4Aa^2)}{8}(t - t_0) \cos \omega(t - t_0) - \tfrac{A^3}{32\omega} \sin 3\omega(t - t_0). \tag{3.67}$$

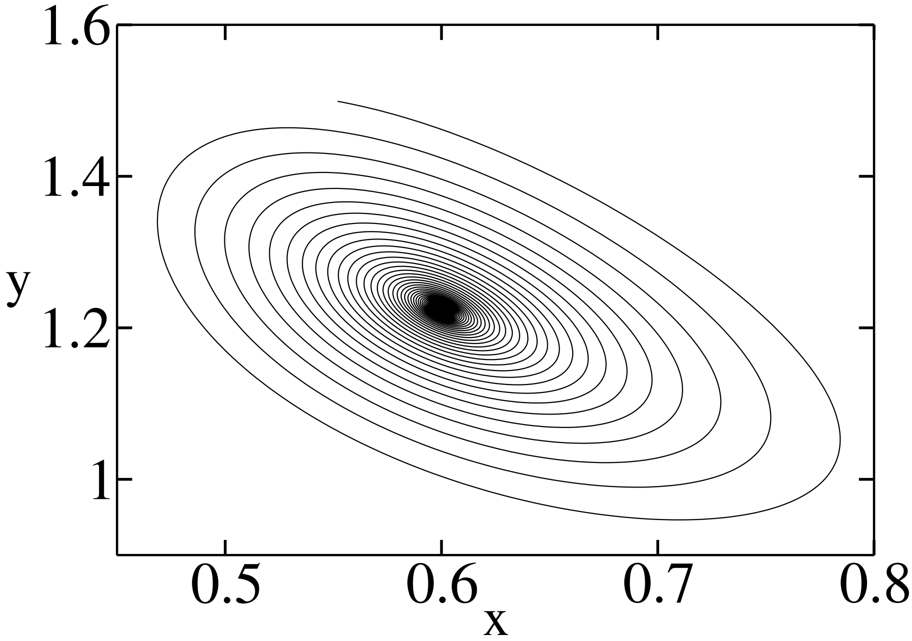

Figure 3.7: *Glycolytic oscillator:* Phase portrait of (3.45) when parameters are chosen in such a way that $F(0,0) > 0$ i.e. a=0.13 and b=0.6 lie on stable focus.

Then $x(t)$ becomes on using $\theta_0 = -\omega t_0$,

$$x(t) = A\cos(\omega t + \theta_0) + \epsilon\left[\frac{(7A^3 - 16Aa^2)}{32\omega}\sin(\omega t + \theta_0) - \frac{(A^3 - 4Aa^2)}{8}(t - t_0)\cos(\omega t + \theta_0) - \frac{A^3}{32\omega}\sin 3(\omega t + \theta_0)\right]. \tag{3.68}$$

Now considering some perturbation in the time interval, $(t - t_0)$ by splitting $(t - \tau) + (\tau - t_0)$, where $t_0 < \tau < t$ and τ is very close to t_0. We define the interval $(t - \tau)$ as a principal part and the remaining part $(\tau - t_0)$ can be neglected because of smallness.

Suppose that on taking perturbation on the time interval the amplitude and the phase slightly be changed from A to $A(\tau)$ and θ_0 to $\theta(\tau)$. Then from RG technique the relation between them are obtained as $A(\tau) = \frac{A}{Z_1(\tau,t_0)}$ and $\theta(\tau) = \theta_0 - Z_2(\tau, t_0)$ where

$$Z_1(\tau, t_0) = 1 + \sum_1^\infty \epsilon^n p_n \text{ and } Z_2(\tau, t_0) = 0 + \sum_1^\infty \epsilon^n q_n. \tag{3.69}$$

Since we are neglecting $O(\epsilon^2)$ then we obtain,

$$Z_1(\tau, t_0) = 1 + \epsilon p_1 + O(\epsilon^2) \text{ and } Z_2(\tau, t_0) = \epsilon q_1 + O(\epsilon^2). \tag{3.70}$$

Now if we put the functions Z_1 and Z_2 as well as A and θ_0 in (3.68) and remove the terms which could lead to divergence we must get either p_1 is zero or anything containing $(\tau -$

t_0) and the same for q_1. But because of the smallness of $(\tau - t_0)$ we can take p_1 and q_1 approximately to zero. So after using the above consideration the constant A becomes $A(\tau)$ and constant θ_0 becomes $\theta(\tau)$ i.e. they become dependent upon the time variable, τ. Again if any term is multiplied directly by $(t - t_0)$ in the final solution of $x(t)$ then it can be converted to $(t - \tau)$ by neglecting the non-principal part. Here a term is present which is directly multiplied with $(t - t_0)$, so neglecting it and using all above results in Eq. (3.68) we get,

$$
\begin{aligned}
x(t) = A(\tau)\cos(\omega t + \theta(\tau)) + \epsilon \Bigg[& \frac{\{7A^3(\tau) - 16A(\tau)a^2\}}{32\omega}\sin(\omega t + \theta(\tau)) - \\
& \frac{\{A^3(\tau) - 4A(\tau)a^2\}}{8}(t - \tau)\cos(\omega t + \theta(\tau)) - \frac{A^3(\tau)}{32\omega}\sin 3(\omega t + \theta(\tau))\Bigg].
\end{aligned}
\tag{3.71}
$$

Finally $\left(\frac{\partial x}{\partial \tau}\right)|_t = 0$ gives,

$$
\begin{aligned}
&\Bigg[\cos(\omega t + \theta(\tau)) + \epsilon\{\frac{(21A^2(\tau) - 16a^2)}{32\omega}\sin(\omega t + \theta(\tau)) \\
&- \frac{(3A^2(\tau) - 4a^2)}{8}(t - \tau)\cos(\omega t + \theta(\tau)) - \frac{3A^2(\tau)}{32\omega}\sin 3(\omega t + \theta(\tau))\}\Bigg]\frac{dA}{d\tau} \\
&+ \Bigg[-A(\tau)\sin(\omega t + \theta(\tau)) + \epsilon\{\frac{(7A^3(\tau) - 16a^2 A(\tau))}{32\omega}\cos(\omega t + \theta(\tau)) \\
&\quad + \frac{(A^3(\tau) - 4a^2 A(\tau))}{8}(t - \tau)\sin(\omega t + \theta(\tau)) - \frac{3A^3(\tau)}{32\omega}\cos 3(\omega t + \theta(\tau))\}\Bigg]\frac{d\theta}{d\tau} = \\
&\qquad -\epsilon\frac{(A^3(\tau) - 4a^2 A(\tau))}{8}\cos(\omega t + \theta(\tau))
\end{aligned}
\tag{3.72}
$$

which leads to

$$
\begin{aligned}
\frac{dA}{d\tau} &= \frac{\epsilon A(\tau)}{2}\{a^2 - \frac{A^2(\tau)}{4}\} \text{ and} \\
\frac{d\theta}{d\tau} &= 0.
\end{aligned}
$$

Now, if we convert the limit cycle condition, $-\epsilon a^2 < 0$ to $-\epsilon a^2 = 0$ i.e. $a = 0$ then it reduces to $\frac{dA}{d\tau} = -\frac{\epsilon A^3(\tau)}{8}$ and $\frac{d\theta}{d\tau} = 0$, showing an isochronous center-type solution. It also satisfies the solution of Van der Pol oscillation, $a = 1$ analyzed in [49]. For $a = 0$, the oscillatory equation also reduces to a center-type as $\epsilon \to 0$ in the Van der Pol type system.

Fig. 3.8 shows a stable limit cycle oscillation when $a = 0.5$ and Fig. 3.9 shows a center-type oscillation when a is fixed at zero. So we obtain a center-type oscillator which is isochronous. Thus for $F(0,0) = 0$, limit cycle breaks down and it gives an isochronous oscillator. It is found numerically that a limit cycle can be obtained in this system for generalized Van der Pol system, $a \in \mathbb{R}_{\neq 0}$ when $0 < \epsilon a^2 < 8.14$ satisfies.

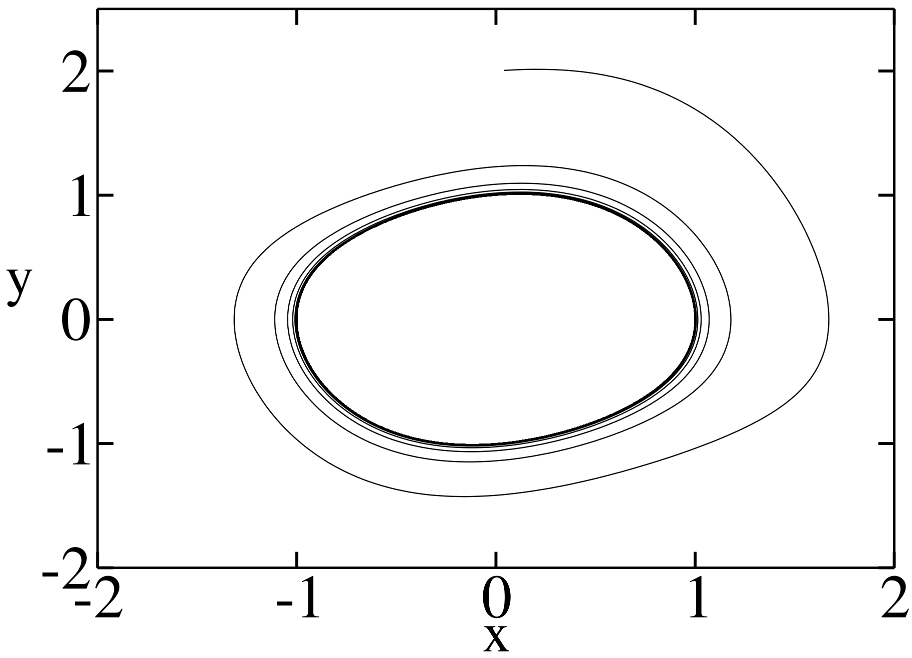

Figure 3.8: *Van der Pol oscillator:* Phase portrait of (3.63) when $a = 0.5$ gives a stable limit cycle for $F(0,0) < 0$.

3.5 CONCLUSION

By casting a class of chemical oscillations usually governed by two-variable kinetic equations into the form of a Liénard oscillator here we have found the conditions of limit cycle and isochronicity. It is shown that the conditions are dictated by the nonlinear damping coefficient and the potential or the forcing term which can be controlled by the suitable choice of the experimental parameters of the chemical oscillators. Although the conditions of limit cycle and isochronicity are shown here with two variables, this mathematical method along with its numerical applicability can also be important for real higher order system. More specifically the main findings in this work are as follows.

1. When the two-dimensional kinetic equations are transformed into a Liénard system of equation the condition for limit cycle and isochronicity can be stated in a unified way. In terms of the Liénard-type oscillator, the condition for limit cycle is given by $F(0,0) < 0$ whereas for the condition of satisfying an isochronous oscillator is $F(0,0) = 0$.

2. When the limit cycle condition i.e. $F(0,0) < 0$ modifies to its boundary i.e. $F(0,0) = 0$ depending on the suitable choice of parameters, the Liénard-type oscillator transforms into an isochronous oscillator.

3. For any Liénard system, when it converts into an isochronous oscillator the system looses its limit cycle stability and it becomes of center-type.

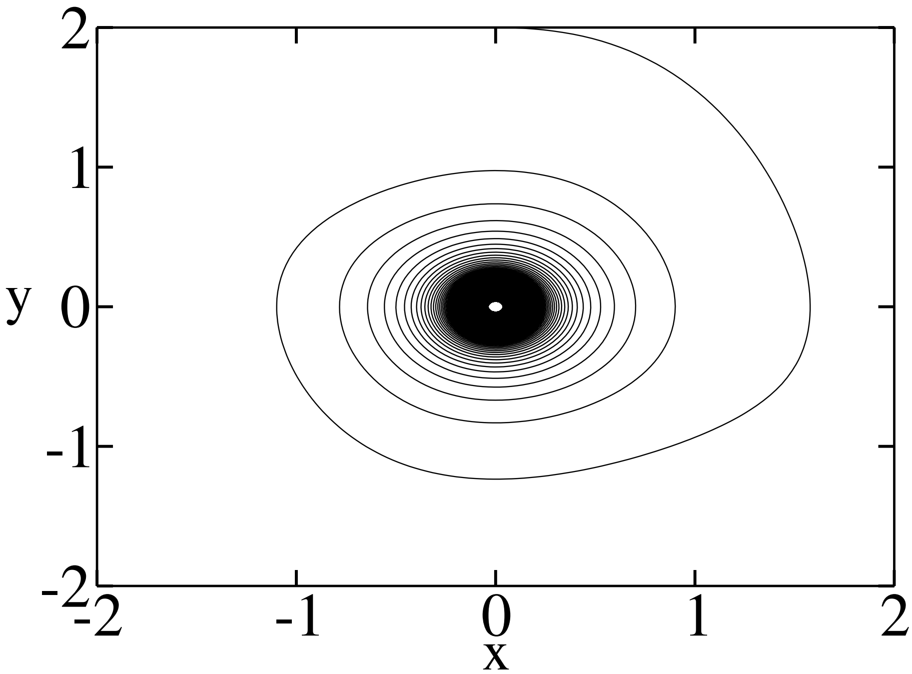

Figure 3.9: *Van der Pol oscillator:* Phase space diagram of (3.63) when $a = 0$ satisfies $F(0, 0) = 0$ which leads to a center.

4

WHEN AN OSCILLATING CENTER IN AN OPEN SYSTEM UNDERGOES POWER LAW DECAY

4.1 INTRODUCTION

Dynamical systems[3, 10, 34, 54, 57, 61, 62] capable of having isochronous oscillations[48, 64, 80, 81] are very important from the point of view of modelling real-world systems[40, 48, 64] which exhibit self-sustained oscillation[11, 12, 63, 65]. The chemical oscillations[33, 36, 50, 74] are also of immense importance in biological world to maintain a cyclic steady state e.g., Glycolytic oscillations[28, 77–79], Calcium oscillations[52], cell division[82], Circadian oscillation[51, 83] and others[36]. Although a lot of work has been performed in finding ways to determine if a system has a limit cycle, surprisingly a little is known about how to find this and still it remains a highly active area of research[10, 28, 36, 80]. To obtain the nonlinear dynamical features of a periodic orbit the general trend is to resort to a geometrical approach coupled with tools of analysis[3, 34, 54, 57, 61, 62]. Recently Renormalisation Group (RG) analysis[46, 47] is heavily used to probe the multi-scale oscillation in the nonlinear system.

A class of arbitrary autonomous kinetic equations in two variables are cast into the form of a Liénard–Levinson–Smith (LLS) oscillator[34, 40, 48, 49, 64] characterized by the nonlinear forcing and damping coefficients which can provide a unified approach to many problems concerning the existence of limit cycle and center. Here[1] our focus is to characterize the properties of periodic orbits to distinguish among center and slowly decaying center-type oscillation using an approximate solution of a class of two variable equations from a multi-scale perturbation theory by Krylov–Bogolyubov (K-B) approach[34, 54]. Although there is a condition which can distinguish limit cycle and center by using a constant part of damping coefficient in LLS equation but still explicit dynamical behaviour of their differences are not very well understood. We have shown the condition in which a center undergoes a slowly decaying orbit and their long time behaviour. Here in addition to the usual geometric disposition of periodic orbits with the shape and size in the steady state we have explored

1 Some portion of this chapter is published in the J. Math. Chem.-Saha et al. (2018)

their asymptotic dynamics in terms of the energy consumption per cycle to distinguish these types of periodic orbits which can appear in diverse situations.

In section (4.2), we have formulated the problem in terms of LLS oscillator. In section (4.3) taking examples of various systems we have studied the dynamical consequences of limit cycle, center and slowly decaying center-type in (a) Glycolytic oscillator, (b) Lotka-Volterra model, (c) Van der Pol type oscillator and (d) time delayed nonlinear feedback oscillator. In section (4.4), we have studied all the above examples analytically and numerically and the main dynamical features of limit cycle, center and slowly decaying center-type are summarized in a table. In later section (4.5), we have explored the source of slow decay of the center. The chapter[2] is concluded in section (4.6).

4.2 APPROXIMATE SOLUTION OF LIÉNARD–LEVINSON–SMITH (LLS) SYSTEM: DESCRIPTION OF THE PROBLEM

Let us consider a two-dimensional set of equations for open system,

$$\begin{aligned} \frac{dx}{dt} &= a_0 + a_1 x + a_2 y + f(x,y), \\ \frac{dy}{dt} &= b_0 + b_1 x + b_2 y + g(x,y), \end{aligned} \tag{4.1}$$

where x and y are populations of two intermediate species of a dynamical process with a_i and b_i are all real parameters expressed in terms of the kinetic constants for all $i = 0, 1, 2$ with $f(x,y)$ and $g(x,y)$ are nonlinear functions for which x_s and y_s are the steady state values.

Then we define a new pair of variables, (z, u) as

$$\begin{aligned} u &= \alpha_0 + \alpha_1 x + \alpha_2 y, \\ z &= \beta_0 + \beta_1 x + \beta_2 y, \end{aligned} \tag{4.2}$$

where $\alpha_0, \alpha_1, \alpha_2$ and $\beta_0, \beta_1, \beta_2$ are constants expressed in terms of a_i and b_i. Now considering u and z in such a way that,

$$\frac{dz}{dt} = u, \tag{4.3}$$

one can obtain an equation of $\ddot{z}$ from the equation of u. Now, using the steady state value $z_s = x_s + y_s$ one can find the LLS oscillator[10, 34, 40, 48, 64] for deviation from the stationary point from z i.e. $\xi(= z - z_s)$ as

$$\ddot{\xi} + F(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0 \tag{4.4}$$

2 Some portion of this chapter is published in the J. Math. Chem.- Saha et al. (2018)

where, $F(\xi,\dot{\xi})$ is the damping function and $G(\xi) = \omega^2\xi + O(\xi)$, is an odd polynomial. Let us take $m = |F(0,0)|$ and rescaling the damping force function by $F_1(\xi,\dot{\xi})$ such that $F(\xi,\dot{\xi}) = mF_1(\xi,\dot{\xi})$, LLS equation can be rewritten as

$$\ddot{\xi} + mF_1(\xi,\dot{\xi})\dot{\xi} + G(\xi) = 0. \tag{4.5}$$

An open system where oscillation is not straight forwardly obvious one can cast the two-dimensional equations into LLS oscillator form which is amenable to multi-scale perturbation analysis. Note that the nature of the defining quantity of limit cycle and center is given by $F_1(0,0) = j$, say. For limit cycle solution, $j = -1$, for asymptotic solution, $j = +1$, and $j = 0$ if the nature of the solution of the system is center which means for the center case there is no time independent damping part. In both center and slowly decaying center-type orbits one can find, $j = 0$, so center and slowly decaying center-type orbits can not be distinguished from the LLS equation form. From the linear stability analysis also one can not distinguish center from slowly decaying center-type cases as, $F(0,0) = -2$ (Real part of eigenvalue) $= 0$. The question is what is the source of this slow decay?

Now rescaling the time variable t by τ with $\tau = \omega t$ and $\xi(t)$ changing to $Z(\tau)$, one can obtain the form of a weakly nonlinear oscillator

$$\ddot{Z}(\tau) + \epsilon h(Z(\tau),\dot{Z}(\tau)) + Z(\tau) = 0, \tag{4.6}$$

where $\epsilon = \frac{m}{\omega^2}$ and $h(Z,\dot{Z})$ may contain nonlinear damping term in LLS oscillator or explicitly time dependent terms for non-autonomous system such as time delayed oscillator and the control parameter ϵ must lie in between 0 and 1 and more so if $0 < m \ll \omega^2$, to have valid perturbative expansion. Applying K-B with a running average $\overline{U}(\tau) = \frac{\varsigma}{2\pi}\int_0^{\frac{2\pi}{\varsigma}} U(s)ds$ over each cycle with ς as the natural frequency of the system(4.6), one can obtain,

$$\begin{aligned}
\dot{\bar{r}} &= \langle \epsilon\, h(Z,\dot{Z})\ \sin(\tau+\phi(\tau))\rangle_\tau = \varphi_1(\bar{r},\bar{\phi}), \\
\dot{\bar{\phi}} &= \langle \frac{\epsilon\, h(Z,\dot{Z})}{r(\tau)} \cos(\tau+\phi(\tau))\rangle_\tau = \varphi_2(\bar{r},\bar{\phi}),
\end{aligned} \tag{4.7}$$

where $Z(\tau) = r(\tau)\ \cos(\tau+\phi(\tau))$. Since $\dot{r}(\tau)$ and $\dot{\phi}(\tau)$ are of $O(\epsilon)$ then we may set the perturbation on r and ϕ over one cycle as, $r(\tau) = \bar{r} + O(\epsilon)$ and $\phi(\tau) = \bar{\phi} + O(\epsilon)$. The functions φ_1 and φ_2 can be obtained from the explicit form of h for the particular cases in the next section. The above system is in a coupled form but most of the cases one can get decoupled set of equations so called amplitude and phase equations otherwise it will be quite harder to solve which is as good as the original system of equations; so it needs to be solved by further approximation to get the amplitude equation. From K-B approach one can define the system energy, $E = \frac{Z^2+\dot{Z}^2}{2}$ along with the energy consumption per cycle, $\Delta E = \frac{d}{d\tau}(\pi\bar{r}^2)$, where $O(\epsilon^2)$ terms are neglected.

4.3 SOME OPEN SYSTEMS AND THEIR COMPARATIVE GENERIC FEATURES

Here we have studied a few prototypical examples to study the dynamical nature of limit cycle, center and slowly decaying center-type periodic orbits. The first class of examples are from autonomous kinetic processes which are important in biology and chemistry, namely Glycolytic oscillator[6, 27, 77–79], Lotka-Volterra model[10, 28, 36, 80] and a slightly generalized version of Van der Pol type oscillator[10, 28, 87] which can be converted to Leinard form. Next we have considered a non-autonomous system, a delay induced feedback model of nonlinear oscillator with a cubic nonlinearity. In the delay model we can find both limit cycle, center and slowly decaying center-type oscillators in different parameter range.

4.3.1 *Glycolytic oscillator*

The Glycolytic oscillator[6, 27, 77–79] is found in the glycolysis by yeast, which can be described by the overall dynamics and biochemical properties of its enzyme phospho-fructokinase. Kinetic properties are simplified by the mathematical analysis of Sel'kov[6, 27] and related models[77]. Basic equations of a Glycolytic oscillation is given by

$$\begin{aligned} \dot{x}(t) &= -x + (a + x^2)y, \\ \dot{y}(t) &= b - (a + x^2)y, \end{aligned} \tag{4.8}$$

where x and y are the intermediate species concentrations with $(x_s = b, y_s = \frac{b}{a+b^2})$ as the fixed point. Considering, the above system in LLS form[87] and using the above procedure in section 4.2, one can find the approximate solution of the above equation as

$$\begin{aligned} x(t) &= b + \omega\bar{r}\sin(\omega t + \bar{\phi}), \\ y(t) &= \frac{b}{a + b^2} + \bar{r}\{\cos(\omega t + \bar{\phi}) - \omega\sin(\omega t + \bar{\phi})\} \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} \dot{\bar{r}} &= -\frac{\epsilon\bar{r}}{8}\left(\frac{3\,\omega^2}{m}\,\bar{r}^2 + 4j\right), \\ \dot{\bar{\phi}} &= \frac{\bar{r}^2}{8} \end{aligned} \tag{4.10}$$

along with the approximate system energy $E = \frac{1}{2}\left[\{x(t) + y(t) - z_s\}^2 + \frac{\{b - x(t)\}^2}{\omega^2}\right]$. Now three cases may arise as in this example, j can assume any of its three values. The condition of having a stable limit cycle is given by $j = -1$, with the radius of the cycle $\frac{2}{\omega}\sqrt{\frac{m}{3}}$ at $t \to \infty$. Next, the condition of having a stable fixed point is with $j = 1$ and in the asymptotic limit radius goes to zero exponentially. And finally, the condition of having a decaying center-type solution for $j = 0$, where the radius decays as a power law with asymptotic expression, $\bar{r} \propto t^{-\frac{1}{2}}$.

4.3.2 *Lotka-Volterra model*

Lotka-Volterra equation is the basic prey predator model[10, 28, 36, 80] which is known to have an oscillatory solution and this oscillation is shown to be have a center. The prey population is x and the predator population is y with the dynamical equation is given by,

$$\begin{aligned} \dot{x}(t) &= \alpha x - \beta x y, \\ \dot{y}(t) &= -\gamma y + \delta x y, \end{aligned} \tag{4.11}$$

with $\alpha, \beta, \gamma, \delta > 0$. The two fixed points are $(x_s = 0, y_s = 0)$ and $(x_s = \frac{\gamma}{\delta}, y_s = \frac{\alpha}{\beta})$ where the first one is a saddle point and the later one gives a center solution obtained from linear stability analysis. The LLS form of the above system (4.11) is given in Appendix B. The approximate analytical solution takes the form,

$$\begin{aligned} x(t) &= \frac{\gamma}{\delta} + \frac{\bar{r}}{z_s \delta}\{\gamma \cos(\omega t + \bar{\phi}) - \omega \sin(\omega t + \bar{\phi})\}, \\ y(t) &= \frac{\alpha}{\beta} + \frac{\bar{r}}{z_s \beta}\{\alpha \cos(\omega t + \bar{\phi}) + \omega \sin(\omega t + \bar{\phi})\}, \end{aligned} \tag{4.12}$$

and $E = \frac{1}{2}\left[\{\delta x(t) + \beta y(t) - z_s\}^2 + \frac{\{\alpha\delta x(t) - \beta\gamma y(t)\}^2}{\omega^2}\right]$. Here $\dot{\bar{r}} = 0$ and $\dot{\bar{\phi}} = 0$, indicates no correction in amplitude as well as phase are needed.

4.3.3 *A generalised Van der Pol oscillator*

We consider here a little generalised form of Van der Pol Oscillator[10, 28, 87] instead of considering $a = 1$ which is traditionally used. This generalised system can provide both a limit cycle, and a slowly decaying center-type solution by changing the parameter, a, unlike the traditional Van der Pol oscillator with only a Limit cycle solution with non-zero value of ϵ_1. Basic equations are given by,

$$\begin{aligned}\dot{x}(t) &= y,\\ \dot{y}(t) &= -\epsilon_1 y(x^2 - a^2) - \omega^2 x,\end{aligned} \tag{4.13}$$

where $0 < \epsilon_1 \ll 1$ is a small perturbative constant and $(x_s = 0, y_s = 0)$ is the only fixed point.

Considering LLS form[10, 87], finally one can obtain the approximate analytical solution of the form,

$$\begin{aligned}x(t) &= \bar{r}\cos\left(\omega t + \bar{\phi}\right),\\ y(t) &= -\omega\bar{r}\sin\left(\omega t + \bar{\phi}\right),\end{aligned} \tag{4.14}$$

where,

$$\begin{aligned}\dot{\bar{r}} &= -\frac{\epsilon\bar{r}}{8a^2}\left(\bar{r}^2 - 4a^2\right),\\ \dot{\bar{\phi}} &= 0,\end{aligned} \tag{4.15}$$

and $E = \frac{1}{2}\left(x^2 + \frac{y^2}{\omega^2}\right)$ with $\epsilon = \frac{\epsilon_1 a^2}{\omega}$.

For this system two cases can arise, a limit cycle and a slowly decaying center-type solution. Condition of having a stable limit cycle for a real positive value of a, $\dot{\bar{r}}$ becomes, $\dot{\bar{r}} = -\frac{\epsilon\bar{r}}{8a^2}\left(\bar{r}^2 - 4a^2\right)$ which gives the radius of the limit cycle is $2a$ as $t \to \infty$. Again, the condition of having a slowly decaying center-type solution can be obtained for $a = 0$ which gives, $\dot{\bar{r}} = -\frac{\epsilon_1}{8\,\omega}\,\bar{r}^3$ and it gives a power law decay as $t \to \infty$ with $\bar{r} \propto t^{-\frac{1}{2}}$.

4.3.4 *Time delayed nonlinear feedback oscillator*

Many nonlinear dynamical systems in various scientific disciplines are influenced by the finite propagation time of signals in feedback loops modelled with a time delay[49, 261]. In some systems, such as lasers and electro mechanical models, a large variety of delays appear. We have provided a time delayed model to obtain both limit cycle, center and slowly decaying center-type[49, 87] oscillation for different range of parameters of the system. There is no general scheme to handle delay system using perturbation theory[262] to study bifurcation or periodic orbit and characterization of the properties of oscillation in limit cycle, center and slowly decaying center-type cases[49].

First we consider here a model of delay system where the oscillation is fed energy through a delay term and its total energy increases with time. We have introduced a linear damping term to make a center solution and then in presence of damping and delay when we introduce another quartic nonlinear term[74, 261] in the potential it gives a limit cycle and slowly decaying center-type solution by tuning the parameters. The basic equations of the model are

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -\epsilon\{x(t-t_d) + \dot{x}(ax^2+b)\} - \omega^2 x(t), \end{aligned} \tag{4.16}$$

where we have considered, $0 < \epsilon \ll 1$ is a small perturbative parameter and $(0,0)$ is the only fixed point. So, it becomes a delayed Van der Pol system which induces energy into the system. Using K-B averaging scheme one can have the analytical solution

$$\begin{aligned} x(t) &= \bar{r}\cos\big(\omega t + \overline{\phi}\big), \\ y(t) &= -\omega\bar{r}\sin\big(\omega t + \overline{\phi}\big), \end{aligned} \tag{4.17}$$

where,

$$\begin{aligned} \dot{\bar{r}} &= -\frac{\epsilon\bar{r}}{8}\{a\bar{r}^2 - 4\left(\frac{\sin\omega t_d}{\omega} - b\right)\}, \\ \dot{\overline{\phi}} &= \frac{\epsilon}{2\omega}\cos(\omega t_d), \end{aligned} \tag{4.18}$$

and $E = \frac{\omega^2 x^2 + \dot{x}^2}{2}$ assuming $0 < t_d \ll 1$.

4.4 NUMERICAL RESULTS FOR VARIOUS OPEN SYSTEMS

Here we have numerically explored the characteristics of periodic orbits of limit cycle, center and slowly decaying center-type cases in various physical systems namely, Glycolytic, Lokta-Volterra and Van der Pol type oscillator and a delayed nonlinear feedback oscillator. We have shown the validity of approximate amplitude equations in terms of the phase space and the role of phase space dynamics. In these model systems we have shown the characteristic features of limit cycle, center and slowly decaying center-type orbits through their asymptotic approach to steady state in terms of a scaled radius and average energy consumption per cycle.

As a starting example we consider Glycolytic Oscillator where for the different values of the constants, a, b can provide limit cycle or a slowly decaying center-type orbit. At first we discuss for the limit cycle case in this system which arises for $a = 0.11$, $b = 0.6$ and then slowly decaying center-type case with $a = 0$, $b = 1$ with the initial condition(IC) of $x = 0.55$ and $y = 1.45$. Fig. 4.1 shows the phase space portrait of (a) a stable limit cycle and (b) slowly decaying center-type oscillation where the dotted lines indicates the numerical simulation

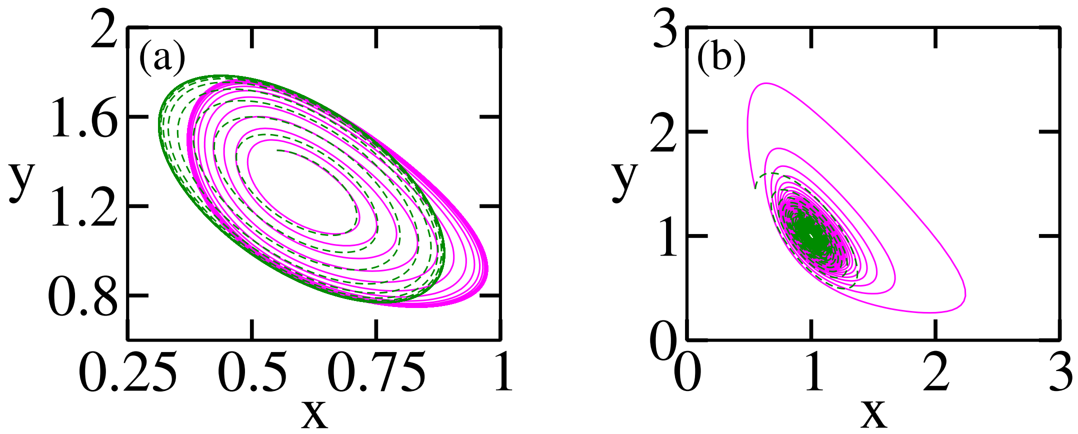

Figure 4.1: *Glycolytic oscillator:* (a) Limit cycle phase portrait with $a = 0.11$ and $b = 0.6$ and (b) slowly decaying center-type phase portrait with $a = 0$ and $b = 1$, where dotted line indicates the numerical simulation of the approximate analytical solution and the solid one is the exact numerical solution of the system.

for the approximate solution and the solid one indicates the exact numerical solution. From the Fig. 4.1(a) and Fig. 4.1(b), it is clear that the approximate solution shows the same nature as the exact one and both gives the limit cycle solution except a phase lag. Reason can be found in the non-zero value of $\dot{\bar{\phi}}$ which brings a phase lag.

In 4.2(a-b) we have shown the dynamics of the scaled radius and energy consumption per cycle of the limit cycle. In 4.2(c-d) similar features of the dynamics of scaled radius and energy change per cycle of the slowly decaying center are shown. From Fig. 4.2(a) and Fig. 4.2(c) we can say that in both limit cycle and slowly decaying center-type cases change of energy per cycle must be zero in the long time limit, however, for the limit cycle it is passed through a maximum as the IC is taken inside the limit-cycle. From Fig. 4.2(b) and Fig. 4.2(d) the scaled radius in limit cycle and slowly decaying center-type cases approach to steady state in a different manner. For limit cycle it becomes a constant and for slowly decaying center it goes as power law decay as $\bar{r}(t) \approx t^{-\frac{1}{2}}$. Fig. 4.2(d) gives a power law fitted curve $r(t) = A_0(1 + A_1 t)^{-0.5}$ with $A_0 = 0.449517$ and $A_1 = 0.151655$.

In the next section we have analyzed the Lotka-Volterra equation with $\alpha = 1.3, \beta = 0.5, \gamma = 0.7, \delta = 1.6$ and $\epsilon = 0.1$ with initial values $x_0 = 0.5$ and $y_0 = 2.5$. Fig. 4.3 shows a phase portrait for the case of center of Lotka-Volterra system with the exact numerical(solid) and approximate solution with amplitude equation(dashed). The dynamics of the scaled radius is a constant from the initial time. Average energy consumption per cycle is zero from the initial time for the case of center which is quite different from the case of limit cycle and slowly decaying center-type orbit. The features of scaled radius and average energy per cycle for the center is attributed to the fact that the initial point is always on the orbit for the case of a center. Since here $\dot{\bar{\phi}} = 0$, there does not exist any phase lag between the exact and

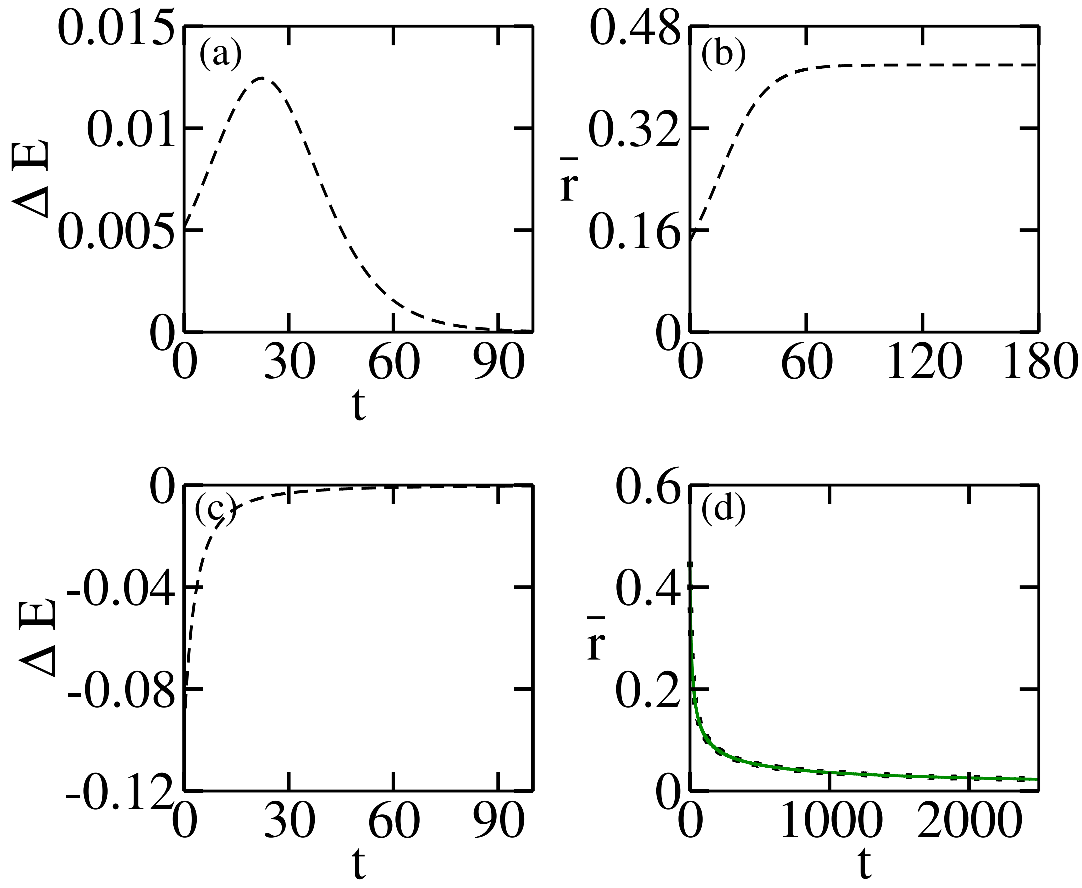

Figure 4.2: *Glycolytic oscillator:* For the limit cycle case(a-b) when the IC is inside the orbit: (a) energy consumption per cycle goes to zero in the steady state after passing through a maximum; (b) scaled radius of the limit cycle is shown. For the case of slowly decaying center-type: (c)energy consumption per cycle starts with a very small negative value to reach zero as time increases (d) the scaled radius decreases with a power law decay where the dotted one is the fitting curve.

approximate centers shown in Fig. 4.3 which also reflects the energy change per cycle with time which is zero as $\dot{\bar{r}} = 0 \implies \bar{r} = r_0$ and depends on the initial value which defines the radius of the center.

In the next section we have analyzed a little generalized version of Van der Pol equation where a is a parameter unlike usual Van der Pol case($a = 1$) with $a = 0.5$ for the limit cycle and $a = 0$ for slowly decaying center-type case with $\epsilon = 0.5$ and IC, $(x_0 = 2, y_0 = 0)$ is taken from outside of the cycle. Since here we find $\dot{\bar{\phi}} = 0$ then there does not exist any phase lag between the two cycles where it is shown in Fig. 4.4 (phase portrait) and Fig. 4.5 shows scaled radius and the energy consumption per cycle with time. In Fig. 4.5(a-b) shows the energy consumption per cycle and scaled radius with time as expected from the limit cycle case. Energy consumption per cycle would pass through a maximum if the IC would be inside the orbit as in the Glycolytic case which is not shown in figure. In Fig. 4.5(c-d) it is

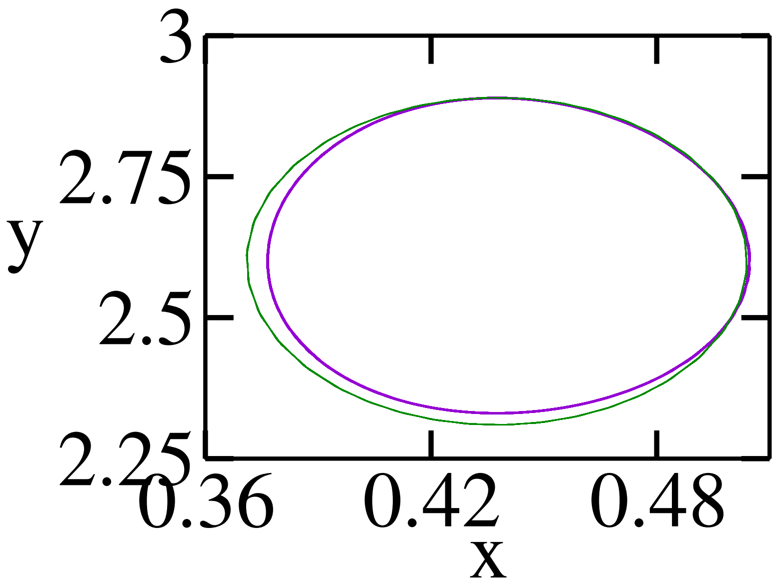

Figure 4.3: *Lotka-Volterra:* Phase space graph of the center with $\alpha = 1.3, \beta = 0.5, \gamma = 0.7, \delta = 1.6$ and $\epsilon = 0.1$, where exact and approximate curves have no phase lag.

the slowly decaying center-type case which shows the energy consumption per cycle gives the same behaviour as in 4.5(a) but the scaled radius with time gives a power law decay with a fitted curve, $r(t) = A_0 + A_1(1 + A_2 t)^{-0.5}$ with $A_0 = 0.000167043$, $A_1 = 1.99394$ and $A_2 = 0.0498935$.

For the delay model with no nonlinearity and damping, i.e, $a = 0, b = 0$ is a simple feedback oscillator with continuously increasing energy in the system. The phase portraits are shown in Fig. 4.6 and Fig. 4.7 with a finite delay, $t_d = 0.623$ with taking the parameters, $\epsilon = 0.05$ and $\omega = 1$ and the initial value $x_0 = 1.5$ and $y_0 = 0.5$. Fig. 4.6(b) shows the phase space effect of delay (outer curve) in respect of non-delay (inner curve) in 3D with time. As in this case both $\dot{\bar{r}} \neq 0$ and $\dot{\bar{\phi}} \neq 0$ so there may exist a phase lag between the two phase space plots and the lag increases with time delay.

In Fig. 4.7 we have shown the phase space curves in (a) feed back system with no damping and nonlinearity where area is increasing, 4.7(b) a phase space for the center, 4.7(c) a limit cycle and in 4.7(d) a slowly decaying center-type orbit for different parameters of a and b.

In Fig. 4.8 we have shown the time dependent nature of the scaled radius for different parameters of a and b. In purely feedback system with no damping and nonlinearity almost exponentially increasing radius in (a), 4.8(b) a radius is a constant from the initial time for the center, 4.8(c) radius changes from its inital value to reach a constant corresponding to a limit cycle and in 4.8(d) radius decreases slowly with power law decay for the case of a slowly decaying center-type orbit.

In Fig. 4.9 we have shown the time dependent nature of the average energy consumption per cycle(ΔE) for different parameters of a and b. For feedback system with no damping

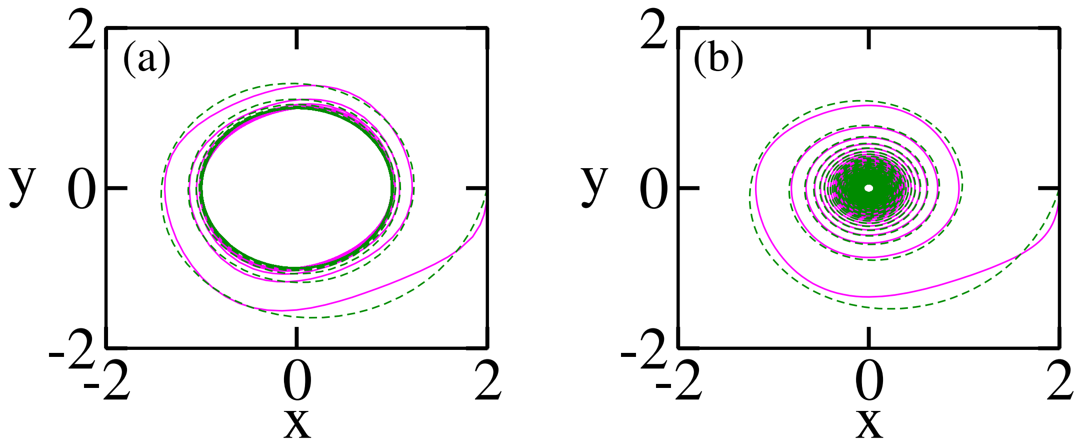

Figure 4.4: *Van der Pol type oscillator:* (a) Limit cycle phase portrait for $a = 0.5$ and (b) slowly decaying center-type phase portrait for $a = 0$ where ϵ is fixed with 0.5 in both cases. The dotted lines indicates the numerical simulation of the approximate analytical solution and the exact numerical solution of the system in the solid line.

and nonlinearity almost exponentially increasing ΔE in (a), 4.9(b) ΔE is a constant from the initial time for the center, 4.9(c) ΔE changes from its inital value to its vanishing value corresponding to a limit cycle and in 4.9(d) ΔE goes to zero from its initial value for a slowly decaying center-type orbit.

The results of the numerical exploration are summarized in Table 4.1 to illustrate the dynamical features of limit cycle, center and slowly decaying center-type orbits. The multi-scale perturbation theory is adopted here from the Lecture notes of Strogatz which is based on the K-B averaging method. It is applied for the LLS system and a non-autonomous system of delayed nonlinear feedback model. The scaled amplitude equation generates all the results of shape, size of the limit cycle, center and slowly decaying center-type orbits almost exactly except in few limit cycle cases a phase lag is found which needs the equation of phase also to be solved simultaneously. The amplitude equation can also be utilized as a stability criteria where the nature of the periodic orbit can be explicitly assigned. This method is giving comparable result with RG method which we have not shown here but checked analytically and numerically. However, it can not give correct result for strongly nonlinear cases.

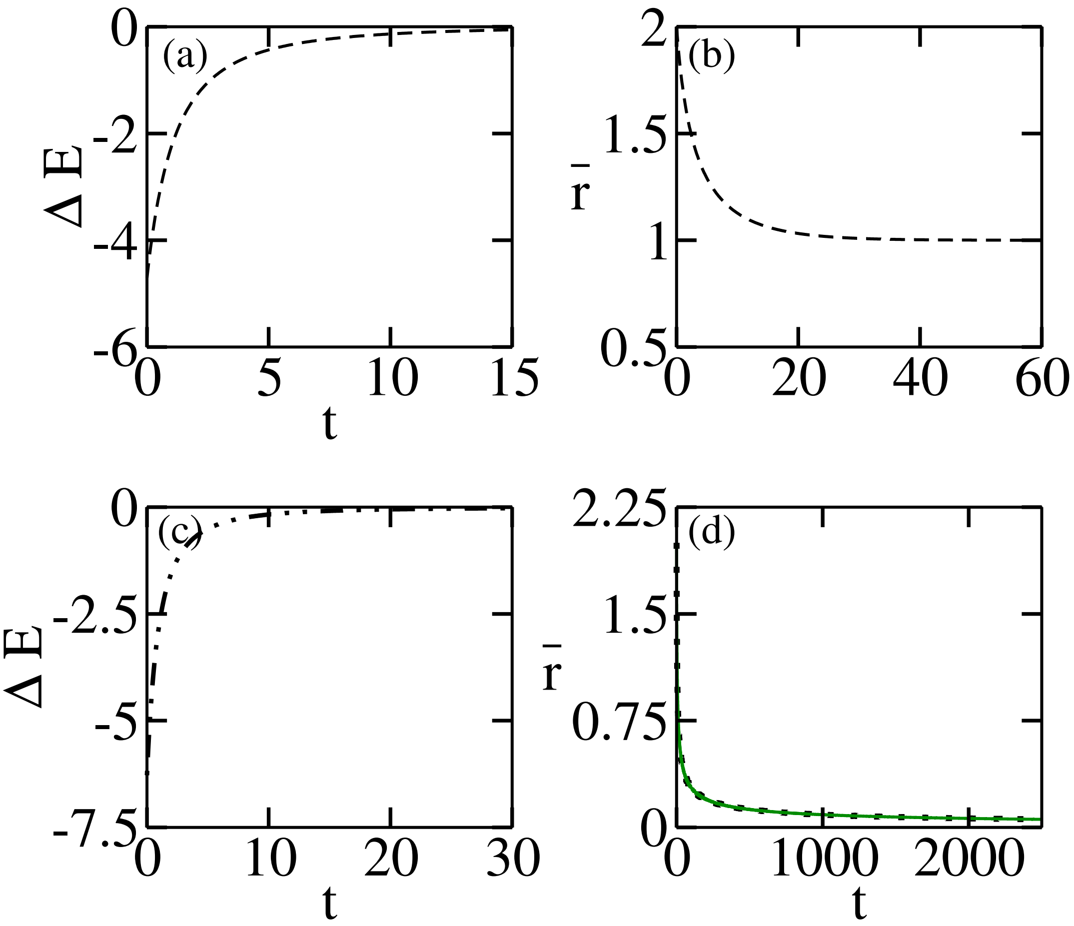

Figure 4.5: *Van der Pol type oscillator:* For the limit cycle case(a-b) when the IC is outside the orbit: (a) energy consumption per cycle starting from a negative value it goes to zero in the steady state; (b) scaled radius of the limit cycle assumes a constant value in the steady state. For the case of slowly decaying center-type: (c)energy consumption per cycle starts with a very small negative value to reach zero as time increases (d)the scaled radius decreases with a power law decay where the dotted one is the fitting curve.

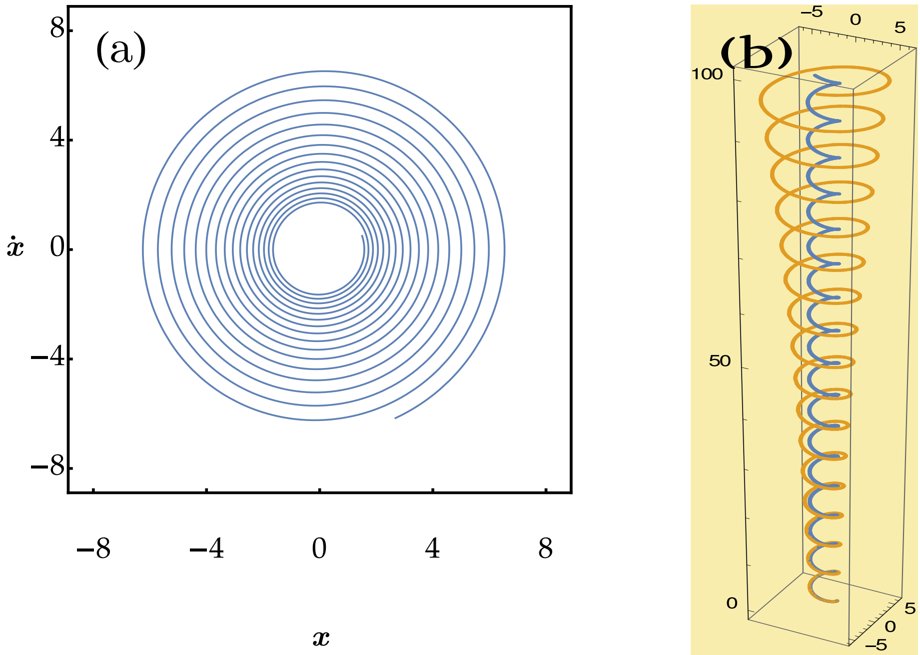

Figure 4.6: *Time-delayed system:* (a) Phase portrait of delay induced feedback oscillator with no damping and nonlinearity, $a = 0, b = 0$ with $t_d = 0.623$ (directly calculated in Mathematica) (b) 3D phase space plot of delay (outer curve) in respect of non-delay (inner curve) with time.

4.5 SOURCE OF POWER LAW DECAY

We have shown in various two-dimensional open systems where the slowly decaying center undergoes a power law decay with exponent $\frac{1}{2}$. The question is how a center undergoes a power law decay? The source of power law decay can be traced in the nonlinear damping function in the LLS system.

It is very straight forward to show that the restoring force will not affect the amplitude equation. To show that we restrict ourselves to the case of LLS equation where the $F(\xi, \dot{\xi})$ and $G(\xi)$ are the polynomial functions of ξ and $\dot{\xi}$. It is well known that linear functional forms of F and G preclude the existence of center. This can be readily seen by considering the typical examples, e.g., a harmonic oscillator or a weakly nonlinear oscillator with a potential $\frac{1}{2}\omega^2 x^2 + \frac{1}{3}\lambda x^4$, $0 < \lambda < 1$ or a Lotka-Volterra model, where one can encounter a center.

In LLS system $F(\xi, \dot{\xi})$ and $G(\xi)$ can be polynomial function of ξ and $\dot{\xi}$ and depending on the even-odd properties of $F(\xi, \dot{\xi})$ and $G(\xi)$ amplitude and phase of the oscillation affect the center and limit cycle of the system. In what follows we employ K-B method of averaging to show that the characteristic even and odd powers of polynomials play crucial role in determining the behaviour of the associated amplitude and phase equations.

Unlike a harmonic oscillator which has a center solution, for a nonlinear damping case, $\ddot{x} + \omega^2 x = -\lambda x^3, 0 < \lambda \ll 1$, one can find a decaying center solution[49]. Next, considering

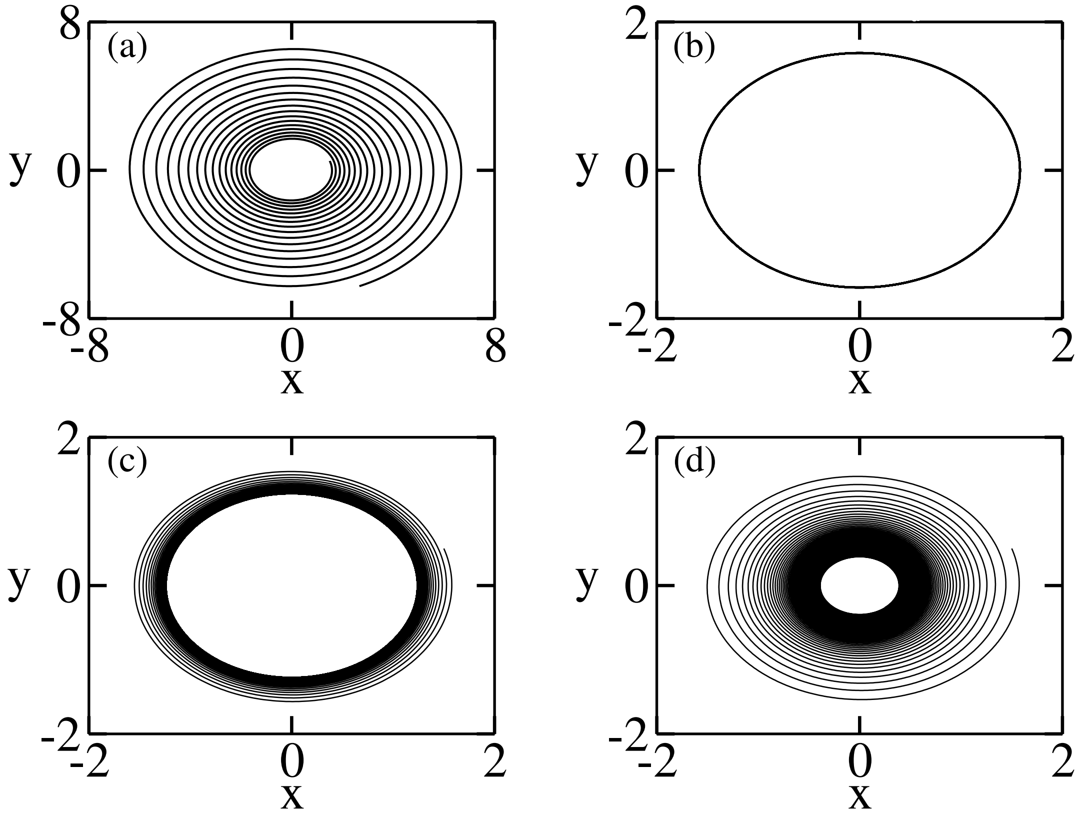

Figure 4.7: *Time-delayed system:* Phase space plots of the numerical simulation of the approximate amplitude equation with the same time delay for (a) $a = 0, b = 0$ one gets a feedback system with increasing phase space area, (b) a center with $a = 0, b = \frac{Sin(\omega t_d)}{\omega}$, (c) a limit cycle with $a = 1, b < \frac{Sin(\omega t_d)}{\omega}$ and (d) a slowly decaying center-type orbit with $a = 1, b = \frac{Sin(\omega t_d)}{\omega}$

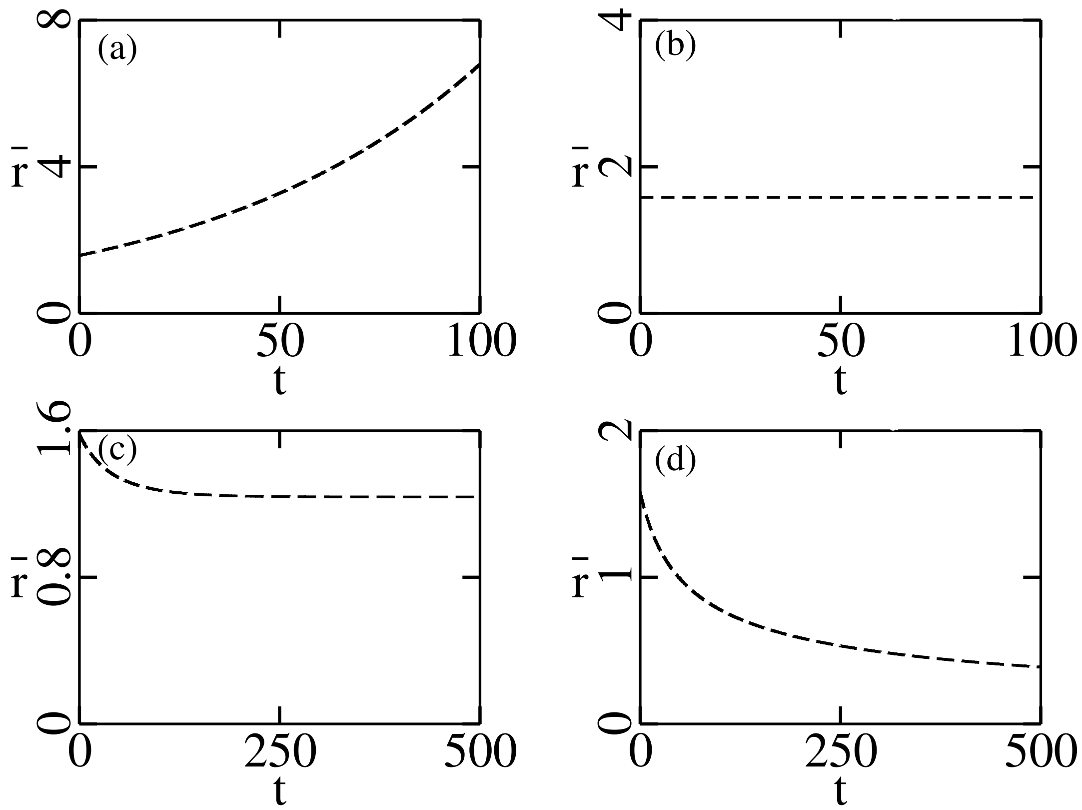

Figure 4.8: *Time-delayed system:* Scaled radius is shown as a function of time in (a) increasing exponentially for purely feedback case, in (b) a constant from the initial time for the center, in (c) changes from its IC outside the cycle to reach a constant corresponding to a limit cycle and in (d) decreases slowly with power law decay for the case of a slowly decaying center-type orbit.

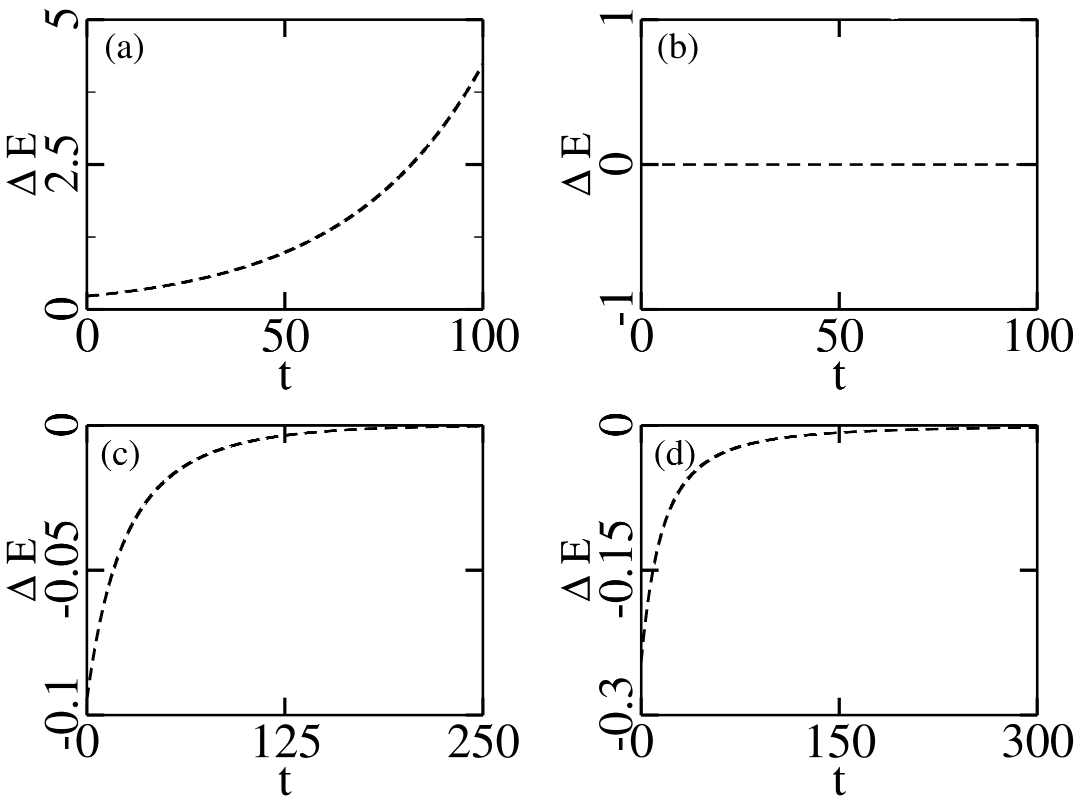

Figure 4.9: *Time-delayed system:* The average energy consumption per cycle(ΔE) is shown as a function of time in (a) increasing exponentially for purely feedback case, in (b) a constant from the initial time for the center, in (c) changes from its IC out side the cycle to reach zero corresponding to a limit cycle and in (d) goes to zero for the case of a slowly decaying center-type orbit.

Periodic Orbits	**Steady State Value (SSV) / Approach to Steady State (ASS) whether IC is inside or outside**	**Limit Cycle**	**Center**	**slowly decaying center-type orbit**
Average scaled radius ($\overline{r}$)	•**SSV** •**ASS**	A constant independent of Initial Condition (IC) **(a)** When IC is inside it increases and saturates to a fixed value **(b)** When IC is out side it decreases and saturates to a fixed value	A constant value depending on IC As IC is always on the orbit, it becomes fixed from initial time	Asymptotically decreasing very slowly by following Power Law ($\propto t^{-\frac{1}{2}}$) relaxation Same as SSV
Average energy consumption per cycle ($\Delta E = 2\pi\overline{r}\,\dot{\overline{r}}$)	•**SSV** •**ASS**	Zero **(a)** When IC is inside it initially increases within a very short region and passing through a peak then gradually falls to zero **(b)** When IC is outside it increases from negative value to zero	Zero Fixed from initial time	Zero From a negative value increases to zero
$\dot{\overline{\phi}}$	•**SSV** •**ASS**	Fixed value (Zero/ Non-zero) **(a)** When IC is inside it initially increases and then saturates to a fixed value or Zero **(b)** When IC is outside it decreases and saturates to a fixed value or Zero	Fixed value (Zero/ Non-zero) Fixed value (Zero/ Non-zero)	Almost Zero Almost Zero
$\dot{\overline{r}}$	•**SSV** •**ASS**	Zero **(a)** When IC is inside it initially increases within a very short region and attains a maximum before going to zero **(b)** When IC is outside it increases from a negative value to zero	Zero Zero throughout all time	Zero Increases from a negative value to zero

Table 4.1: Dynamical features of limit cycle, center and decaying center-type orbits

both damping and nonlinear restoring force, a typical example of Lotka-Volterra system with LLS form (see Appendix B), $\ddot{x} + \epsilon_1(b_1 x + b_2\dot{x})\dot{x} + \omega^2 x + \epsilon_1 b_3 x^2 = 0, 0 < \epsilon_1 \ll 1$, gives a center solution. Here, the damping force function is of linear order and also odd as $F(\xi, \dot{\xi}) \neq F(-\xi, -\dot{\xi})$. From the various examples, it is not clear when a center undergoes power law decay. So we would like to introduce a little more general scenario when it appears.

To begin with we consider polynomial functions of $F(\xi, \dot{\xi})$ and $G(\xi)$ with upto cubic power of $\dot{\xi}$ and ξ in the following reduced forms,

$$\begin{aligned} F(\xi, \dot{\xi}) &= -[A_{01} + A_{11}\xi + A_{21}\xi^2 + A_{31}\xi^3 + A_{02}\dot{\xi} + A_{12}\xi\dot{\xi} + A_{22}\xi^2\dot{\xi} + A_{32}\xi^3\dot{\xi} \\ &\quad + A_{03}\dot{\xi}^2 + A_{13}\xi\dot{\xi}^2 + A_{23}\xi^2\dot{\xi}^2 + A_{33}\xi^3\dot{\xi}^2], \\ G(\xi) &= -[A_{10}\xi + A_{20}\xi^2 + A_{30}\xi^3]. \end{aligned} \tag{4.19}$$

Let us take $|F(0,0)| = \sigma \in \mathbb{R}^+$, an arbitrary constant with $F(\xi, \dot{\xi}) = \sigma F_\sigma(\xi, \dot{\xi})$. Then the LLS equation can be rewritten as

$$\ddot{\xi} + \sigma F_\sigma(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0. \tag{4.20}$$

Therefore the final equation takes the form of a nonlinear oscillator after rescaling t by τ taking, $\omega t \to \tau$ as

$$\ddot{Z}(\tau) + \epsilon h(Z(\tau), \dot{Z}(\tau)) + Z(\tau) = 0, \tag{4.21}$$

where, $0 < \epsilon = \frac{\sigma}{\omega^2} \ll 1$, $\omega^2 = -A_{10} > 0$ and $Z(\tau) \equiv \xi(t)$ and $\dot{Z}(\tau) \equiv \dot{\xi}(t)$. Eq. (4.21) is amenable to averaging with K-B method which gives

$$\begin{aligned} h(Z, \dot{Z}) = -\Big[\{&B_{01} + B_{11}Z + B_{21}Z^2 + B_{31}Z^3 + B_{02}\omega\dot{Z} + B_{12}Z\omega\dot{Z} + B_{22}Z^2\omega\dot{Z} + B_{32}Z^3\omega\dot{Z} + B_{03}\omega^2\dot{Z}^2 \\ &+ B_{13}Z\omega^2\dot{Z}^2 + B_{23}Z^2\omega^2\dot{Z}^2 + B_{33}Z^3\omega^2\dot{Z}^2\}\omega\dot{Z} + B_{20}Z^2 + B_{30}Z^3\Big], \end{aligned} \tag{4.22}$$

where $B_{ij} = \frac{A_{ij}}{\sigma}$, $i, j = 1, 2, 3$ are the corresponding indices. The amplitude and phase equations are obtained as,

$$\begin{aligned} \dot{\bar{r}} &= \frac{\epsilon\omega\bar{r}}{16}\{\bar{r}^2\left(B_{23}\bar{r}^2\omega^2 + 6B_{03}\omega^2 + 2B_{21}\right) + 8B_{01}\} + O(\epsilon^2), \\ \dot{\bar{\phi}} &= -\frac{\epsilon\bar{r}^2}{16}\left(B_{32}\bar{r}^2\omega^2 + 2B_{12}\omega^2 + 6B_{30}\right) + O(\epsilon^2). \end{aligned} \tag{4.23}$$

Now from a detailed analysis of the amplitude equation for $\dot{\bar{r}}$, it is apparent that only even elements of $F(\xi, \dot{\xi})$ appears but none of the elements of $G(\xi)$ is present due to the vanishing value of the averages of $\sin^\mu \cos^\nu$ terms with $\mu = 1$ and $\nu \in \mathbb{Z}$. The non-zero averages arise only when μ, ν both are even i.e. $\mu = 2\eta_1, \nu = 2\eta_2; \eta_1, \eta_2 \in \mathbb{Z}$. The power law solution of amplitude with $t^{-\frac{1}{2}}$ can appear from Eq. (4.23) only when the right hand side contains r^3

term which means when A_{23} or B_{23} terms should be absent and there must be non-zero positive value of any one of the terms, A_{03} and A_{21} should be present.

4.6 CONCLUSIONS

By suitably adopting K-B averaging method in multi-scale perturbation theory for a periodic system here we have provided the solution of a class of two variable open systems through LLS form. The approximate K-B solution is shown to be almost exact for calculating physical properties with a set of diverse examples, namely, Glycolytic oscillator, Lotka-Volterra system, a generalized Van der Pol oscillator and a time delayed nonlinear feedback oscillator. To characterize a slowly decaying center-type oscillator one can find that even when the constant part of damping force in LLS equation vanishes i.e, $F(0,0) = 0$, the center may undergo a slow decay and asymptotically gives a stable focus and the radius of the center undergoes a power-law decay. Here we have investigated about the source of this power law decay and in this context we have compared the asymptotic dynamics of the limit cycle, center and slow decay of center-type orbit in various open systems and a generic feature of all these systems are explored.

The condition of isochronicity which is usually defined as the amplitude independent period of the orbit is utilized to characterize the dynamical features of limit cycle, center and slowly decaying center-type orbits as pointed below. While the shape of the orbits are shown in phase space of actual variables, the size of the orbit is studied by introducing an average scaled radius variable and asymptotic approach of the orbits to steady state can be understood from the scaled radius and energy consumption per cycle.

1. From the approximate solution we can find the limit cycle, center or slowly decaying center-type motion almost exactly just by using the single variable equation of the radius, so called amplitude equation. However, to get rid of the slight phase lag in the limit cycle case one needs to solve the equation of phase variable coupled with radius variable simultaneously.

2. The energy consumption per cycle ΔE, of the limit cycle vanishes at the steady state depending on the position of the initial condition which can be inside or outside of the cycle in a particular way: (a) when the initial condition is inside the cycle, it increases initially to a maximum in a very short time and then goes to zero, (b) when the initial condition is outside the cycle, ΔE increases from a negative value to zero. For the case of a center quite distinctly it is zero from the initial time upto the steady state. For a slowly decaying center, ΔE increases from a negative value to zero.

3. Most interesting feature about the difference in center and slowly decaying center-type oscillation, which is indistinguishable from the LLS equation form, i.e., $F(0, 0) = 0$, the slowly decaying center-type case reveals a power law decay of the radius, $t^{-\frac{1}{2}}$ asymptotically unlike the center where the radius becomes almost constant from initial time, however, ΔE for slowly decaying center-type oscillation arrives at zero in a finite time.

5

REDUCTION OF KINETIC EQUATIONS TO LIÉNARD–LEVINSON–SMITH (LLS) FORM: COUNTING LIMIT CYCLES

5.1 INTRODUCTION

Various open kinetic systems[10–12, 28, 36, 44] in physics, chemistry and biology, are generically described by a minimal model of autonomous coupled differential equations[3, 34, 54, 59, 60, 84] of two variables. A Rayleigh[17] equation in violin string and Van der Pol oscillation in electric circuit are the classic examples, in this context. More generally, Liénard[10, 39–43] equation underlines the concrete criteria for the existence of at least one limit cycle for a general class of such systems of which Van der Pol is a special case of the form $\ddot{x} + f(x)\dot{x} + x = 0$ where $f(x) = \epsilon(x^2 - 1)$ and Liénard transformation is $\dot{x} = y - F(x)$ and $\dot{y} = -x$ with $F(x) = \int_0^x f(\tau)d\tau$. A further generalisation of Liénard equation is the Liénard–Levinson–Smith (LLS) equation[41–44], $\ddot{x} + F(x, \dot{x})\dot{x} + G(x) = 0$, sometimes called the generalised Liénard equation. Casting a general system of kinetic equations in two variables which describe a variety of scenarios in physical, chemical, biochemical and ecological sciences into LLS form[44] is often not straight-forward[10, 40, 44, 87]. To this end we have provided a scheme for a wide class of open nonlinear equations, cast in the LLS form so that the later becomes amenable to several techniques in nonlinear dynamics.

Our next objective[1] is to find the nature and the number of limit cycles for a given LLS equation thereby addressing the second part of the Hilbert's 16^{th} problem. The problem of counting limit cycle has a long legacy since Hilbert, Smale and many others and still continues it without complete understanding[3, 39, 84, 255, 256, 259, 263, 264]. Our scheme is based on the Krylov–Bogolyubov (K-B) method of averaging[10, 34, 54, 265], a variant of multi-scale perturbation technique[10, 46, 47, 80] to derive amplitude equation with considering the polynomial forms of the nonlinear damping and restoring force functions. We have illustrated our results on a variety of known model systems[39, 264, 266] with single and multiple limit cycles[39, 264, 266].

1 Some portion of this chapter is published in the Int. J. Appl. Comp. Math.-Saha et al. (2019)

In section (5.2), the problem have been formulated in terms of LLS oscillator to apply perturbation theory where little a bit shorter version is provided to the reduction of kinetic equations to LLS form. In section (5.3) perturbation theory is applied to find maximum number of limit cycles for a LLS system. In section (5.4), we have reviewed various model system starting from one cycle cases to k-cycle cases to establish a connection with the cycle counting hypothesis. The chapter[2] is finally concluded in section (5.5).

5.2 REDUCTION OF KINETIC EQUATIONS TO LIÉNARD–LEVINSON–SMITH (LLS) FORM: CONDITIONS FOR LIMIT CYCLE

We consider here a set of two-dimensional autonomous kinetic equations for an open system. Our aim is to cast the equations into a form of a variant of LLS oscillator[10, 40, 87] or LLS oscillator[10, 34, 40, 44, 87] which can further be reduced to Rayleigh and Liénard form. Let us begin with the system of autonomous kinetic equations

$$\begin{aligned} \frac{dx}{dt} &= a_0 + a_1 x + a_2 y + f(x,y), \\ \frac{dy}{dt} &= b_0 + b_1 x + b_2 y + g(x,y), \end{aligned} \tag{5.1}$$

where $x(t)$ and $y(t)$ are, for example, field variables or populations of species of chemical, biological or ecological process [10–12, 28] with a_i, b_i for $i = 0,1,2$ are all real parameters expressed in terms of the appropriate kinetic constants. Let, (x_s, y_s) be the fixed point of the system and $f(x,y)$ and $g(x,y)$ are the nonlinear functions of x and y. A first step is shifting the steady state (x_s, y_s) to the origin $(0,0)$ with the help of a linear transformation as LLS system is a second order homogeneous ordinary differential equation (ODE).

The linear transformation can be chosen by introducing a new pair of variables (ξ, u), both of which are functions of x and y where $\xi = \beta_0 + \beta_1 x + \beta_2 y$ with $\beta_0 = -(\beta_1 x_s + \beta_2 y_s)$ i.e. $\xi = \beta_1(x - x_s) + \beta_2(y - y_s)$ such that $\dot{\xi} = u$. β_1, β_2 are weighted constants such that it makes the new steady state at the origin, $\xi_s = 0$, $u_s = 0$. u is expressed as $u = \alpha_0 + \alpha_1 x + \alpha_2 y$, with β_i, α_i for $i = 0,1,2$ are all real constants which can be expressed in terms of system parameters. From the inverse transformation we can easily obtain the expressions for x and y as given by

$$\begin{aligned} x &= \tfrac{\alpha_2(\beta_0 - \xi) + \beta_2(u - \alpha_0)}{\alpha_1\beta_2 - \alpha_2\beta_1} = L(\xi, u), \\ y &= \tfrac{\alpha_1(\xi - \beta_0) + \beta_1(\alpha_0 - u)}{\alpha_1\beta_2 - \alpha_2\beta_1} = K(\xi, u), \end{aligned} \tag{5.2}$$

2 Some portion of this chapter is published in the Int. J. Appl. Comp. Math.-Saha et al. (2019)

provided that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. Differentiating again, $\dot{\xi} = u$ with respect to the independent variable t we get,

$$\begin{aligned}
\ddot{\xi} &= \dot{u} = \alpha_1\dot{x} + \alpha_2\dot{y} \\
&= \alpha_1\{a_0 + a_1L(\xi,\dot{\xi}) + a_2K(\xi,\dot{\xi}) + \varphi(\xi,\dot{\xi})\} + \alpha_2\{b_0 + b_1L(\xi,\dot{\xi}) + b_2K(\xi,\dot{\xi}) + \phi(\xi,\dot{\xi})\}, \quad (5.3)
\end{aligned}$$

where, $L(\xi,\dot{\xi}) = c_1\xi + c_2\dot{\xi} + c_L$ and $K(\xi,\dot{\xi}) = c_3\xi + c_4\dot{\xi} + c_K$ with $\begin{bmatrix} c_1 & c_2 & c_L \\ c_3 & c_4 & c_K \end{bmatrix} = \frac{1}{\alpha_1\beta_2-\alpha_2\beta_1}$ $\begin{bmatrix} -\alpha_2 & \beta_2 & \alpha_2\beta_0 - \alpha_0\beta_2 \\ \alpha_1 & -\beta_1 & \alpha_0\beta_1 - \alpha_1\beta_0 \end{bmatrix}$. The functions φ and ϕ can be expressed as a power series expansion as,

$$\varphi(\xi,\dot{\xi}) = \sum_{n,m=0}^{\infty} \varphi_{nm}\xi^n\dot{\xi}^m \text{ and } \phi(\xi,\dot{\xi}) = \sum_{n,m=0}^{\infty} \phi_{nm}\xi^n\dot{\xi}^m, \quad (5.4)$$

with, $\phi(\xi,\dot{\xi}) = \mu\varphi(\xi,\dot{\xi})$, as the functions f and g are related through μ by $g = \mu f$, $\mu \in \mathbb{R}$. So, after putting the above form in equation (5.3) one can find,

$$\begin{aligned}
\ddot{\xi} &= \alpha_1a_0 + \alpha_1a_1(c_1\xi + c_2\dot{\xi} + c_L) + \alpha_1a_2(c_3\xi + c_4\dot{\xi} + c_K) + (\alpha_1 + \mu\alpha_2)\sum_{n,m=0}^{\infty}\varphi_{nm}\xi^n\dot{\xi}^m \\
&\quad + \alpha_2b_0 + \alpha_2b_1(c_1\xi + c_2\dot{\xi} + c_L) + \alpha_2b_2(c_3\xi + c_4\dot{\xi} + c_K), \textit{i.e.}, \\
\ddot{\xi} &= A_{00} + \left(A_{10} + \sum_{n>1}A_{n0}\xi^{n-1}\right)\xi + \left(A_{01} + \sum_{n>0}A_{n1}\xi^n + \sum_{n\geq 0}\sum_{m>1}A_{nm}\xi^n\dot{\xi}^{m-1}\right)\dot{\xi}, \quad (5.5)
\end{aligned}$$

where, $\alpha_1a_0 + \alpha_2b_0 + (\alpha_1 + \mu\alpha_2)\varphi_{00} + (\alpha_1a_1 + \alpha_2b_1)c_L + (\alpha_1a_2 + \alpha_2b_2)c_K = A_{00} = 0$ (by definition of a zero fixed point of ξ), $A_{10} = \alpha_1(a_1c_1 + a_2c_3) + \alpha_2(b_1c_1 + b_2c_3) + (\alpha_1 + \mu\alpha_2)\varphi_{10}$, $A_{01} = \alpha_1(a_1c_2 + a_2c_4) + \alpha_2(b_1c_2 + b_2c_4) + (\alpha_1 + \mu\alpha_2)\varphi_{01}$, $A_{n0} = (\alpha_1 + \mu\alpha_2)\varphi_{n0}$, $A_{n1} = (\alpha_1 + \mu\alpha_2)\varphi_{n1}$ and $A_{nm} = (\alpha_1 + \mu\alpha_2)\varphi_{nm}$, where indices follow the values as given in the summation over $m, n \in \mathbb{Z}^+$. Finally, the above equation looks like,

$$\ddot{\xi} + F(\xi,\dot{\xi})\dot{\xi} + G(\xi) = 0, \quad (5.6)$$

where, the functions $F(\xi,\dot{\xi})$ and $G(\xi)$ are given by

$$\begin{aligned}
F(\xi,\dot{\xi}) &= -\left[A_{01} + \sum_{n>0}A_{n1}\xi^n + \sum_{n\geq 0}\sum_{m>1}A_{nm}\xi^n\dot{\xi}^{m-1}\right], \\
G(\xi) &= -\left[A_{10} + \sum_{n>1}A_{n0}\xi^{n-1}\right]\xi. \quad (5.7)
\end{aligned}$$

Eq. (5.6) is a well known equation of generalised Liénard form called LLS equation. The condition for existence of having at least a locally stable limit cycle of the dynamical system is $F(0,0) < 0 \implies A_{01} > 0$. It can be shown from the linear stability analysis that there is a relation between $F(0,0)$ and eigenvalues $(\lambda_\pm)$ with, $F(0,0) = -2\ Re(\lambda_\pm)$. For a LLS system, there are six conditions to have a limit cycle are given in [40, 42–44]. Out of these six conditions, the condition $F(0,0)$ plays an important role to have a locally stable or unstable limit cycle for such kind of system[34, 40, 44, 87] depending upon the sign of $F(0,0)$ is < 0 or > 0, respectively. In particular, two situations may arise:

I: For $A_{nm} = 0$, with $n \geq 2, \forall m$ i.e. there be an unique steady state ($\xi_s = 0$) with restoring force linear in ξ, then the above form of (5.6) looks like

$$\ddot{\xi} + F_R(\dot{\xi})\dot{\xi} + G_R(\dot{\xi})\xi = 0, \tag{5.8}$$

where,

$$F_R(\dot{\xi}) = -\left[A_{01} + \textstyle\sum_{m>1} A_{0m}\dot{\xi}^{m-1}\right], \quad G_R(\dot{\xi}) = -\left[A_{10} + \textstyle\sum_{m>0} A_{1m}\dot{\xi}^{m}\right], \tag{5.9}$$

which is in the form of generalised Rayleigh oscillator[17], the limit cycle condition modifies to, $F_R(0) < 0$.

II: For $A_{nm} = 0$, with $m \geq 2, \forall n$, which corresponds to Liénard equation with an unique steady state ($\xi_s = 0$). This is of the form

$$\ddot{\xi} + F_L(\xi)\dot{\xi} + G_L(\xi) = 0, \tag{5.10}$$

where,

$$F_L(\xi) = -\left[A_{01} + \textstyle\sum_{n>0} A_{n1}\xi^{n}\right], \quad G_L(\xi) = -\left[A_{10} + \textstyle\sum_{n>1} A_{n0}\xi^{n-1}\right]\xi, \tag{5.11}$$

where the limit cycle condition is $F_L(0) < 0$. We know that, for a Liénard system, the damping force function, $F_L(\xi)$ and the restoring force function, $G_L(\xi)$ are even and odd functions of ξ, respectively.

However, for generalised Liénard or LLS system the odd-even properties of $G(\xi)$ and $F(\xi,\dot{\xi})$ have complex ramifications[44] for practical systems. Here, we have examined the properties with the help of K-B averaging method.

5.3 MAXIMUM NUMBER OF LIMIT CYCLES

We now restrict ourselves to the case of LLS systems where the $F(\xi,\dot{\xi})$ and $G(\xi)$ are the polynomial functions of ξ and $\dot{\xi}$. It is well known that linear functional forms of F and

G preclude the existence of limit cycle. This can be readily seen by considering the typical examples, e.g., a Harmonic oscillator or a weakly nonlinear oscillator with a potential $\frac{1}{2}\omega_0^2 x^2 + \frac{1}{3}\lambda x^4$, $0 < \lambda < 1$ or a Lotka-Volterra model (see Appendix B), where one encounters a center. We therefore consider the polynomial form of nonlinear damping function $F(\xi, \dot{\xi})$ and restoring force function $G(\xi)$ for our analysis of limit cycle. In what follows we employ K-B method of averaging to show that the characteristic even/odd powers of polynomials play crucial role in determining the behaviour of the associated amplitude and phase equations.

To begin with we consider some fixed values of m, n of Eq. (5.7) to truncate the series at M, N, for the highest power of $\dot{\xi}$ and ξ, respectively. For explicit structure of a prototypical example of an amplitude equation we choose upto $M = N = 3$ for illustration. This includes all possible cases for the even and odd nature of $F(\xi, \dot{\xi})$ and $G(\xi)$, respectively. Then the above form of $F(\xi, \dot{\xi})$ and $G(\xi)$ will be in the following reduced forms,

$$\begin{aligned} F(\xi, \dot{\xi}) &= -[A_{01} + A_{11}\xi + A_{21}\xi^2 + A_{31}\xi^3 + A_{02}\dot{\xi} + A_{12}\xi\dot{\xi} + A_{22}\xi^2\dot{\xi} + A_{32}\xi^3\dot{\xi} \\ &\quad + A_{03}\dot{\xi}^2 + A_{13}\xi\dot{\xi}^2 + A_{23}\xi^2\dot{\xi}^2 + A_{33}\xi^3\dot{\xi}^2], \\ G(\xi) &= -[A_{10}\xi + A_{20}\xi^2 + A_{30}\xi^3]. \end{aligned} \tag{5.12}$$

Let us take $|F(0,0)| = \sigma \in \mathbb{R}^+$, an arbitrary constant with $F(\xi, \dot{\xi}) = \sigma F_\sigma(\xi, \dot{\xi})$. Then the LLS equation can be rewritten as

$$\ddot{\xi} + \sigma F_\sigma(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0. \tag{5.13}$$

Therefore the final equation takes the form of a nonlinear oscillator after rescaling t by τ taking, $\omega t \to \tau$ as

$$\ddot{Z}(\tau) + \epsilon h(Z(\tau), \dot{Z}(\tau)) + Z(\tau) = 0, \tag{5.14}$$

where, $0 < \epsilon = \frac{\sigma}{\omega^2} \ll 1$, $\omega^2 = -A_{10} > 0$ and $Z(\tau) \equiv \xi(t)$ and $\omega\dot{Z}(\tau) \equiv \dot{\xi}(t)$. Eq. (5.14) is now ready for the treatment using K-B method with

$$\begin{aligned} h(Z, \dot{Z}) = -\big[& \{B_{01} + B_{11}Z + B_{21}Z^2 + B_{31}Z^3 + B_{02}\omega\dot{Z} + B_{12}Z\omega\dot{Z} + B_{22}Z^2\omega\dot{Z} + \\ & B_{32}Z^3\omega\dot{Z} + B_{03}\omega^2\dot{Z}^2 + B_{13}Z\omega^2\dot{Z}^2 + B_{23}Z^2\omega^2\dot{Z}^2 + B_{33}Z^3\omega^2\dot{Z}^2\}\omega\dot{Z} + B_{20}Z^2 + B_{30}Z^3\big], \end{aligned} \tag{5.15}$$

where $B_{ij} = \frac{A_{ij}}{\sigma}$, $i, j = 0, 1, 2, 3$ with $B_{00} = 0$ and B_{01} will take the fixed value, -1, 0, or 1 depending upon the nature of the fixed point is stable focus, center/center-type or limit cycle, respectively. Now choosing, $Z(\tau) \approx r(\tau)\ \cos(\tau + \phi(\tau))$ as a solution of Eq. (5.14) we have $\dot{Z}(\tau) \approx -r(\tau)\ \sin(\tau + \phi(\tau))$ with slowly varying radius $r(\tau) = \sqrt{Z^2 + \dot{Z}^2}$ and phase $\phi(\tau) = -\tau + tan^{-1}(-\frac{\dot{Z}}{Z})$. The function $h(Z, \dot{Z})$ contains all the nonlinear terms and ϵ is the nonlinearity controlling parameter i.e. one has to satisfy $0 < \sigma \ll \omega^2$. Then one can obtain

$\dot{r}(\tau) = \epsilon h \sin(\tau + \phi(\tau))$ and $\dot{\phi}(\tau) = \frac{\epsilon h}{r(\tau)} \cos(\tau + \phi(\tau))$ i.e. the time derivative of amplitude and phase are of $O(\epsilon)$. So, after taking a running average[10, 44, 54] of a time dependent function U defined as, $\overline{U}(\tau) = \frac{1}{2\pi}\int_0^{2\pi} U(s)ds$, one finds, $\dot{\bar{r}} = \langle \epsilon h \sin(\tau + \phi(\tau))\rangle_\tau$ and $\dot{\bar{\phi}} = \langle \frac{\epsilon h}{r(\tau)} \cos(\tau + \phi(\tau))\rangle_\tau$, which gives,

$$\begin{aligned}
\dot{\bar{r}} &= \frac{\epsilon\omega\bar{r}}{16}\{\bar{r}^2\left(B_{23}\bar{r}^2\omega^2 + 6B_{03}\omega^2 + 2B_{21}\right) + 8B_{01}\} + O(\epsilon^2), \\
\dot{\bar{\phi}} &= -\frac{\epsilon\bar{r}^2}{16}\left(B_{32}\bar{r}^2\omega^2 + 2B_{12}\omega^2 + 6B_{30}\right) + O(\epsilon^2).
\end{aligned} \tag{5.16}$$

Now from a close look at the equation for $\dot{\bar{r}}$, it is apparent that only even elements of $F(\xi, \dot{\xi})$ appears but none of any elements of $G(\xi)$ is present due to the zero averages of $\sin^\mu\cos^\nu$ terms with $\mu = 1$ and $\nu \in \mathbb{Z}$. The non-zero averages arise only when μ, ν both are even i.e. $\mu = 2\eta_1, \nu = 2\eta_2; \eta_1, \eta_2 \in \mathbb{Z}$. Thus, the effect in $\dot{\bar{r}}$ appears only through the even coefficients of $F(\xi, \dot{\xi})$ i.e. by examining the respective variables in the $\dot{\bar{r}}$ equation, we find that only some even coefficients appear for the first order correction. On the other hand $\dot{\bar{\phi}}$ contains only even coefficients of $F(\xi, \dot{\xi})$ which are not in amplitude equation along with odd coefficients of $G(\xi)$ which shows that only odd $G(\xi)$ plays a role here.

So, from the equation of $\dot{\bar{r}}$, one finds that there exist at most 4 non-zero values of $\bar{r}$. If out of the four roots every pair appears as conjugate then there are three possibilities. The cases are, (i) two different sets of complex conjugate roots giving an asymptotically stable solution, (ii) one pair of complex conjugate roots and two real roots of equal magnitude with opposite sign implying a limit cycle solution having only one cycle and (iii) either four real roots of equal magnitude with opposite sign having double multiplicity gives a limit cycle solution with only one cycle or two different sets of real roots of equal magnitude with opposite sign, may give limit cycle solution with two different cycles of different radius. The unique zero values of the roots of $\bar{r}$ gives a center or center-type[166] situation. So, in short, the existence of a non-zero real root will provide the radius of the cycle which will be stable or unstable depending on the $-ve$ or $+ve$ sign of $\frac{d\dot{\bar{r}}}{d\bar{r}}$, at $\bar{r} = \bar{r}_{ss}$ and at $\bar{r}_{ss} = 0$ $\frac{d\dot{\bar{r}}}{d\bar{r}} > 0$ or < 0 gives the nature of the fixed point.

As an example, for Kaiser model[55, 56, 85, 89, 90, 267–269], there exist three limit cycles for a certain range of α, β. So, if we choose the parameters, α and β from the three limit cycle zone then there exist six real roots with three different pairs i.e., three different radii exist according to three cycles. But, slightly away from the three limit cycle zone, there will exist only a pair of real roots with the same magnitude and other four will appear as a complex conjugate pairs and together produces only a stable limit cycle.

Note that, to have a stable limit cycle solution, one condition must be satisfied i.e. $F(0, 0) < 0$. But, it fails to give how many limit cycles the system can admit. According to the root finding algorithm one can guess the maximum number of cycles of a LLS system. The condition

N	M	$N+M$	Max. No. of Non-zero Real Roots (Even)	Max. No. of Cycle(s)
Even	Even	Even	$N+M-2=(N-1)+(M-1)$	$\frac{N+M}{2}-1$
Even	Odd	Odd	$N+M-1=(N)+(M-1)$	$\frac{N+M-1}{2}$
Odd	Even	Odd	$N+M-3=(N-2)+(M-1)$	$\frac{N+M-3}{2}$
Odd	Odd	Even	$N+M-2=(N-1)+(M-1)$	$\frac{N+M}{2}-1$

Table 5.1: Maxumum number of limit cycles for LLS system

$F(0,0) < 0$ plays an important role as a check for the existence of atleast one stable limit cycle. But for 2-cycle situations one can have at first the locally unstable limit cycle before locating the outer stable limit cycle and in this situation $F(0,0) > 0$.

Based on these considerations we have prepared a Table 5.1 illustrating the possible cases for the maximum number of non-zero real roots or the limit cycles. Now if we denote the non-zero real values of $\bar{r}$ as an existence of limit cycles as $\bar{r}$ gives the radius of the cycle where at the same time a pair of conjugate (one $+ve$ and one $-ve$) roots of equal magnitude exists for such kind of LLS systems then out of these two roots, radius will be measured by the magnitude and each distinct magnitude counts the number of cycles. For example, if there exists six roots, say, $(p,-p)$ occurring twice and $(q,-q)$ occurring once then the number of cycles will be 2 of radius p and q, respectively. So, if there are all real roots occurring once, then the number of cycles will be atmost $\frac{N+M-2}{2}$ or $\frac{N+M-1}{2}$ or $\frac{N+M-3}{2}$. For LLS equation with N, M are the maximum power of ξ and $\dot{\xi}$ respectively, we have performed the K-B analysis numerically for $N = 10, M = 10$. The result is given in Table 5.2. For Rayleigh system with $N = 1$, for all $M \geq 1$, maximum number of limit cycle will be $\frac{M-1}{2}$ or $\frac{M-2}{2}$ for odd or even M, respectively. For Liénard system with $M = 1$ for all $N \geq 1$, the maximum number of limit cycle becomes $\frac{N-1}{2}$ or $\frac{N}{2}$ for odd or even N, respectively. The above table is valid for an arbitrary finite polynomials of F and G. For the case of arbitrary infinite polynomial[255, 256, 259] cases maximum number of limit cycles can be stated for finite truncation.

5.4 APPLICATIONS TO SOME MODEL SYSTEMS

Here we have examined three classes of physical models where the analysis of the maximum number of limit cycles holds. This connection with the general model system is discussed with polynomial damping and restoring force function.

5.4.1 *One-cycle cases: Van der Pol oscillator, Glycolytic oscillator, Brusselator model*

Considering the Van der Pol oscillator[10, 11, 36, 54, 85, 87, 97] with equation, $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$ having the weak nonlinearity for $0 < \epsilon \ll 1$ produces a locally stable limit cycle with $F(0,0) < 0$. So, if we compare with the general table we have $N = 2$ and $M = 1$. This gives a condition for a unique stable limit cycle.

Next considering the Liénard form[87, 166] of Glycolytic oscillator[10, 11, 28, 36, 87, 166] as,

$$\ddot{\xi} + \left[(1 + a + 3b^2) - 2b\xi - 2bk - 3b\dot{\xi} + \xi\dot{\xi} + k\dot{\xi} + \dot{\xi}^2\right]\dot{\xi} + (a + b^2)\xi = 0; a, b > 0, k = b + \frac{b}{a + b^2},$$

has a unique stable limit cycle with $F(0,0) < 0$[87] having $N = 1$ and $M = 3$. This gives one limit cycle.

Furthermore, considering the Brusselator model having the Liénard form[28, 40, 87],

$$\ddot{\xi} + \left[-\frac{2a_1\xi}{\alpha} - b + \frac{a_1^2}{\alpha^2} + \alpha - \frac{2a_1\dot{\xi}}{\alpha^2} + \frac{b\dot{\xi}}{a_1} + \frac{\dot{\xi}^2}{\alpha^2} + \frac{\xi\dot{\xi}}{\alpha}\right]\dot{\xi} + \frac{a_1^2\xi}{\alpha} = 0; a_1, b, \alpha > 0$$

gives a unique stable limit cycle with $F(0,0) < 0$[40, 87], where $N = 1$ and $M = 3$ again giving rise to the same situation.

5.4.2 *Two-cycle cases*

We rewrite the Liénard form according to Ref.[39, 266] as, $\dot{x(t)} = y(t) - F(x(t)), \dot{y(t)} = -x(t)$, where $F(x(t))$ is an odd polynomial. After taking derivative it takes the following form $\ddot{x} + F'(x)\dot{x} + x = 0$, where $F'(x) = \frac{\partial F(x)}{\partial x}$ now becomes the form of an even polynomial. For $F(x) = a_1 x + a_2 x^2 + a_3 x^3$, it has been shown[39, 266, 270] that the system allows a unique limit cycle if $a_1 a_3 < 0$, which will be stable if $a_1 < 0$ and unstable if $a_1 > 0$. This corresponds to the table $N = 2, M = 1$. Further extension by Rychkov[271] shows that for $F(x) = (a_1 x + a_3 x^3 + a_5 x^5)$ the number of limit cycle is atmost two. Numerical simulation corroborates this observation when $F(x)$ is chosen as in Ref.[39, 266, 270], $F(x) = 0.32x^5 - \frac{4}{3}x^3 + 0.8x$. For this case the inner one is unstable limit cycle as $F(0,0) = 0.8 > 0$ i.e. a stable fixed point but the outer one is a stable limit cycle. Here, as per Table 5.1 we have $N = 4$ and $M = 1$. The above table thus gives the strategies to find out the number of limit cycles(both stable and unstable) a system can have. On the other hand our analysis by K-B method provides a hint towards a choice of the parameter space for search of real roots of the radial equation.

5.4.3 *Three-cycles case: Kaiser bi-rhythmic model*

Extending Van der Pol oscillator model with a nonlinear function of higher order polynomial, Kaiser[55, 56, 85, 89, 90, 267–269] has described bi-rhythmicity with the nonlinear equation,

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + x = E\cos\Omega t. \tag{5.17}$$

Here, $\alpha, \beta, \mu > 0$ tune the nonlinearity. This is a prototype self-sustained oscillatory system in absence of E and Ω which are the amplitude and the frequency of the external excitation, respectively. The model exhibits an extremely rich bifurcation behaviour and the system actually produces bi-rhythmicity. It has been emphasized that in the undriven case, the model is a multi-limit cycle oscillator and has three limit cycles, two of them are stable and between the two stable limit cycles there is an unstable one which divides the basins of attraction of the two stable cycles. In presence of E and Ω, the above system exhibits some interesting features[55, 85, 89, 90, 267–269]. From Table 5.1 one has $N = 6$ and $M = 1$ with even-odd sub cases, while $E = 0$. Thus, there may have 6 roots for the radial equation if $\mu > 0$ and α, β(controlling parameters of the radii) are chosen from three limit cycle zone($\alpha = 0.144, \beta = 0.005$) and finally, the number of distinct values will be 3 which implies that the system can have atmost three limit cycles (but here it is exactly 3). Further, if we choose $\beta = 0$ for the above undriven Kaiser model with $\alpha = 0.1$, one can have two limit cycles with radii ≈ 2.35 and ≈ 3.80, respectively, of which the smaller one will be stable and the larger one will be unstable.

Note that, if there are odd number of limit cycles, say l, then out of the $l-$cycles, $\frac{l+1}{2}$ will be stable limit cycles and the remaining $\frac{l-1}{2}$ will be unstable limit cycles iff $F(0,0) < 0$. For example, for the Van der Pol oscillator, Glycolytic oscillator, Brusselator model etc. only one limit cycle exists which is stable. For Kaiser model, $l = 3$ and one can observe the situations accordingly. So, for odd number of cycles innermost one will be locally stable.

5.4.4 *k-cycle cases:*

5.4.4.1 *A model with $N = 1$ and $M = 2k + 1$*

For counting the number of limit cycles Gaiko[264] has shown, for a Liénard-type system i.e., LLS equation having the form,

$$\ddot{x} - \left(\mu_1 + \mu_2\dot{x} + \mu_3\dot{x}^2 + \cdots + \mu_{2k}\dot{x}^{2k-1} + \mu_{2k+1}\dot{x}^{2k}\right)\dot{x} + x \quad = 0, \tag{5.18}$$

$\oplus, R$		M									
		1	2	3	4	5	6	7	8	9	10
N	1	2, 0	3, 0	4, 2	5, 2	6, 4	7, 4	8, 6	9, 6	10, 8	11, 8
	2	3, 2	4, 2	5, 4	6, 4	7, 6	8, 6	9, 8	10, 8	11, 10	12, 10
	3	4, 2	5, 2	6, 4	7, 4	8, 6	9, 6	10, 8	11, 8	12, 10	13, 10
	4	5, 4	6, 4	7, 6	8, 6	9, 8	10, 8	11, 10	12, 10	13, 12	14, 12
	5	6, 4	7, 4	8, 6	9, 6	10, 8	11, 8	12, 10	13, 10	14, 12	15, 12
	6	7, 6	8, 6	9, 8	10, 8	11, 10	12, 10	13, 12	14, 12	15, 14	16, 14
	7	8, 6	9, 6	10, 8	11, 8	12, 10	13, 10	14, 12	15, 12	16, 14	17, 14
	8	9, 8	10, 8	11, 10	12, 10	13, 12	14, 12	15, 14	16, 14	17, 16	18, 16
	9	10, 8	11, 8	12, 10	13, 10	14, 12	15, 12	16, 14	17, 14	18, 16	19, 16
	10	11, 10	12, 10	13, 12	14, 12	15, 14	16, 14	17, 16	18, 16	19, 18	20, 18

Table 5.2: Table for highest degree polynomial $N + M(\oplus)$ for LLS system together with maximum number of distinct conjugate roots(R), with $1 \leq N, M \leq 10$

can have atmost k limit cycles if and only if, $\mu_1 > 0$. The result[264] correlates with our result. For any value of $k \in \mathbb{Z}$ it fits the odd-odd case of the general table and accordingly, M and N are $2k + 1$ and 1, respectively, and finally the number of cycles will be atmost $\frac{N+M}{2} - 1 = k$.

5.4.4.2 *A model with $N = 2k$ and $M = 1$*

Blows and Lloyd[39, 272] have stated that "For the Liénard or LLS system $\dot{x} = y - F(x), \dot{y} = -g(x)$ with $g(x) = x$ and $F(x) = a_1 x + a_2 x^2 + \cdots + a_{2k+1} x^{2k+1}$ has at most k local limit cycles and there are coefficients with $a_1, a_3, \ldots, a_{2k+1}$ altering in sign". This can be found from the Table 5.1 with $N = 2k$ and $M = 1$ to give the condition of atmost k limit cycles. For example, taking $k = 3$ with $F(x) = -\epsilon(72x - \frac{392}{3}x^3 + \frac{224}{5}x^5 - \frac{128}{35}x^7)$ has exactly three limit cycles for sufficiently small $\epsilon \neq 0$ which are circles with radii 1, 2 and 3. The above statement nicely corresponds to the Theorem-6, pp-260[39].

Counting the number of limit cycles through Renormalisation Group (RG) method in first order will give similar result which was done by Das et. al.[255, 256, 259] for some models. We have verified similar results for (3,3) polynomial cases for (F,G) functions using RG method which become increasingly very difficult and almost impossible upto (10,10) case than K-B averaging method as tabulated in this work. It is very useful to count the number of limit cycles from the table by just looking at the LLS form. For example, the number of limit cycles of all models in Ref.[255, 256, 259] along with the models in our work can be estimated from our table. The table-I can also be utilized to prepare a model of a desired number of limit cycles in a systematic way.

5.5 CONCLUSIONS

We have presented a scheme to cast a set of a class of coupled nonlinear equations in two variables into a LLS form. By expressing the nonlinear damping and forcing functions as polynomial we have implemented K-B method of averaging to explore the number of admissible limit cycles of the dynamical systems. Our results can be summarised as follows:

1. For a LLS system, the number of limit cycles will be atmost $\frac{N+M-2}{2}$ when N and M degree of the polynomials for damping and restoring force both are even or odd. Again, $\frac{N+M-1}{2}$ cycles can be found when N is even and M is odd and finally, $\frac{N+M-3}{2}$ cycles when N is odd and M is even.

2. For a Liénard system, in particular, the formula of counting the number of limit cycles follows the same with $M = 1$ and $N \in \mathbb{Z}^+$. Also for the generalised Rayleigh situation there occurs a linear restoring force so that N is 1 and $M \in \mathbb{Z}^+$.

3. We have validated our general result with the help of a variety of physical systems with one, two, three upto arbitrary k-cycles.

4. This method stated in our work can also be utilized to prepare a model of a desired number of limit cycles in a systematic way.

6

SYSTEMATIC DESIGNING OF BI-RHYTHMIC AND TRI-RHYTHMIC MODELS IN FAMILIES OF VAN DER POL AND RAYLEIGH OSCILLATORS

6.1 INTRODUCTION

Both the Rayleigh or Van der Pol oscillators can be subsumed into a common form, i.e., Liénard–Levinson–Smith (LLS) oscillator[16, 40–44, 87, 96, 97], so that they can be viewed as the two special cases[96] of LLS system. While the standard Rayleigh or Van der Pol oscillator allows single limit cycle, because of polynomial nature of nonlinear damping force and restoring force functions, LLS system exhibits multi-rhythmicity[98, 99], i.e., one observes the co-existence of multiple limit cycles in the dynamical system. In some biological systems nature utilizes this multi-rhythmicity as models of regulation and in various auto-organisation of cell signalling[28, 98–101]. In a related issue a bi-rhythmic model for Glycolytic oscillation was proposed by Decorly and Goldbeter[102]. The coupling of two cellular oscillations[99] also leads to multi-rhythmicity. By extending Van der Pol oscillator Kaiser had suggested a bi-rhythmic model[55, 85] which has subsequently been used in several occasions[56, 89–92, 103].

In spite of several interesting studies in different contexts as mentioned above, a systematic procedure for constructing a multi-rhythmic model with a desired number of limit cycles is still lacking. Two problems must be clearly distinguished at this juncture. The first one concerns of finding out the maximum number of limit cycles possible for a LLS system. The problem has been addressed in Ref. [255, 256, 259] and also by us[96]. The second one, our focal theme in this chapter[1] is to systematically construct a minimal model with a desired number of limit cycles starting from a LLS system with a single limit cycle. The essential elements for this design is to choose the appropriate forms of polynomial damping and the restoring force functions. A scheme for critical estimation of the associated parameters for the polynomial functions and a smallness parameter is a necessary requirement. This non-

1 Some portion of this chapter is published in the Communications in Nonlinear Science and Numerical Simulation-Saha et al. (2020)

trivial systematization of parameter space allows us to construct the higher order variants of both Van der Pol and Rayleigh oscillators with three, five and higher number of limit cycles. As illustration, we have proposed two cases of Van der Pol family; first one concerns five limit cycles of which three are stable and two are unstable dividing the basins of attractions. Second one is an alternative version of the bi-rhythmic Kaiser system[55, 85]. We have also proposed the bi- and tri-rhythmic models for the Rayleigh family of oscillators which are hitherto unknown to the best of our knowledge in the context of nonlinear oscillators. Our analysis shows while the mono-rhythmic Rayleigh and Van der Pol oscillators are unique, their bi- or tri-rhythmic or higher order variants may assume different forms depending on the nature of the polynomial functions. Our theoretical analysis is corroborated by detailed numerical simulations.

In section (6.2), the counting of number of limit cycles for polynomial damping and restoring force functions of a Liénard–Levinson–Smith (LLS) system is revisited. The generalisation of single-cycle oscillator to multicycle cases is discussed in section (6.3) by reviewing various model systems starting from one cycle cases to k-cycle cases through the classification of two families — one is Van der Pol family (section 6.3.1) and the another one is Rayleigh family (section 6.3.2). Construction of new families of Van der Pol and Rayleigh oscillators with multiple limit cycles (such as bi-rhythmic and tri-rhythmic) and their alternative forms are discussed in section 6.4 through a detailed investigation. Bi-rhythmicity apart from LLS system is discussed in section 6.5 where the full study is given in Appendix C. Section 6.6 describes a complete flow chart to have multi-rhythmic systems along with some concluding remarks.

6.2 Polynomial damping and restoring force functions for Liénard–Levinson–Smith (LLS) system; number of limit cycles

We begin with a class of LLS equation of the following form

$$\ddot{\xi} + F(\xi, \dot{\xi})\dot{\xi} + G(\xi) = 0, \tag{6.1}$$

where, $F(\xi, \dot{\xi})$ and $G(\xi)$ being the polynomial functions as given by

$$\begin{aligned} F(\xi, \dot{\xi}) &= -\left[A_{01} + \sum_{n>0} A_{n1}\xi^n + \sum_{n\geq 0}\sum_{m>1} A_{nm}\xi^n\dot{\xi}^{m-1}\right], \\ G(\xi) &= -\left[A_{10} + \sum_{n>1} A_{n0}\xi^{n-1}\right]\xi. \end{aligned} \tag{6.2}$$

They refer to the nonlinear damping and force functions, respectively. Depending on the several conditions[40–44]on F and G for the existence of at least one locally stable limit cycle for dynamical model, the following two cases can appear:

I : When $A_{nm} = 0$, $n \geq 2, \forall m$, the above form of Eq. (6.1) takes the form,

$$\ddot{\xi} + F_R(\dot{\xi})\dot{\xi} + G_R(\dot{\xi})\xi = 0, \tag{6.3}$$

where,

$$F_R(\dot{\xi}) \;\; = - \left[A_{01} + \textstyle\sum_{m>1} A_{0m}\dot{\xi}^{m-1}\right], \quad G_R(\dot{\xi}) = - \left[A_{10} + \textstyle\sum_{m>0} A_{1m}\dot{\xi}^{m}\right], \tag{6.4}$$

with the steady state ($\xi_s = 0$) for a restoring force which is linear in ξ. It is a form of generalised Rayleigh oscillator[16] where the condition of limit cycle reduces to, $F_R(0) < 0$. One can have $F_R(\dot{\xi}) = \epsilon(\dot{\xi}^2 - 1)$ and $G_R(\dot{\xi}) = 1$ for the special case of Rayleigh oscillator.

II : When $A_{nm} = 0$, $m \geq 2, \forall n$, gives a Liénard equation with the steady state ($\xi_s = 0$). The oscillator form can be written as

$$\ddot{\xi} + F_L(\xi)\dot{\xi} + G_L(\xi) = 0, \tag{6.5}$$

with,

$$F_L(\xi) \;\; = - \left[A_{01} + \textstyle\sum_{n>0} A_{n1}\xi^{n}\right], \quad G_L(\xi) = - \left[A_{10} + \textstyle\sum_{n>1} A_{n0}\xi^{n-1}\right]\xi, \tag{6.6}$$

where the condition of limit cycle is $F_L(0) < 0$. It is to be noted that for a Liénard system $F_L(\xi)$ and $G_L(\xi)$ are even and odd functions of ξ, respectively. With the special case of Van der Pol oscillator we have $F_L(\xi) = \epsilon(\xi^2 - 1)$ and $G_L(\xi) = \xi$.

Eq. (6.1) is the starting point of our analysis. In a recent communication it has been shown that by implementing Krylov–Bogolyubov (K-B) method [34, 54, 86, 96, 265] one can estimate the maximum number of limit cycles admissible by a dynamical system. Here the basic idea is to introduce the scaled time $\omega t \to \tau$ and the transformed variables $\xi \to Z, \dot{\xi} \to \omega\dot{Z}$ so that $Z(\tau)$ and $\dot{Z}(\tau)$ can be expressed as $Z(\tau) \approx r(\tau)\,\cos(\tau + \phi(\tau))$ and $\dot{Z}(\tau) \approx -r(\tau)\,\sin(\tau + \phi(\tau))$. K-B averages leads us to the equations for average amplitude $\bar{r}$ and average phase $\bar{\phi}$. To proceed further we first terminate the series upto the value, M, N, as the highest power of $\dot{\xi}$ and ξ, respectively. For an amplitude equation we take $M = N$ upto 3. It covers all possible even and odd functions of $F(\xi,\dot{\xi})$ and $G(\xi)$, respectively. Such forms of $F(\xi,\dot{\xi})$ and $G(\xi)$ are reduced to the forms given below,

$$\begin{aligned} F(\xi,\dot{\xi}) &= -[A_{01}+A_{11}\xi+A_{21}\xi^2+A_{31}\xi^3+A_{02}\dot{\xi}+A_{12}\xi\dot{\xi}+A_{22}\xi^2\dot{\xi}+A_{32}\xi^3\dot{\xi} \\ &\quad +A_{03}\dot{\xi}^2+A_{13}\xi\dot{\xi}^2+A_{23}\xi^2\dot{\xi}^2+A_{33}\xi^3\dot{\xi}^2], \\ G(\xi) &= -[A_{10}\xi+A_{20}\xi^2+A_{30}\xi^3]. \end{aligned} \tag{6.7}$$

Abbreviating $|F(0,0)|=\sigma\in\mathbb{R}^+$, with $F(\xi,\dot{\xi})=\sigma F_\sigma(\xi,\dot{\xi})$, the LLS equation after rescaling can be rewritten as

$$\ddot{Z}(\tau)+\epsilon\, h(Z(\tau),\dot{Z}(\tau))+Z(\tau) = 0, \tag{6.8}$$

where, $0<\epsilon=\frac{\sigma}{\omega^2}\ll 1$, $\omega^2=-A_{10}>0$ and $Z(\tau)\equiv\xi(t)$ with $\omega\dot{Z}(\tau)\equiv\dot{\xi}(t)$ and h can be expressed as

$$h(Z,\dot{Z})=-\left[\{H_1+H_2+H_3\}\omega\dot{Z}+B_{20}Z^2+B_{30}Z^3\right], \tag{6.9}$$

where, $H_1=B_{01}+B_{11}Z+B_{21}Z^2+B_{31}Z^3$, $H_2=B_{02}\omega\dot{Z}+B_{12}Z\omega\dot{Z}+B_{22}Z^2\omega\dot{Z}+B_{32}Z^3\omega\dot{Z}$, $H_3=B_{03}\omega^2\dot{Z}^2+B_{13}Z\omega^2\dot{Z}^2+B_{23}Z^2\omega^2\dot{Z}^2+B_{33}Z^3\omega^2\dot{Z}^2$ and $B_{ij}=\frac{A_{ij}}{\sigma}$, $i,j=0,1,2,3$ are the corresponding indices with $B_{0,0}=0$ and B_{01} takes the values (-1, 0, 1) depending on the property of the fixed point (asymptotically stable, center, limit cycle), respectively. K-B averaging yields the following amplitude and phase equations,

$$\begin{aligned} \dot{\bar{r}} &= \frac{\epsilon\omega\bar{r}}{16}\{\bar{r}^2\left(B_{23}\bar{r}^2\omega^2+6B_{03}\omega^2+2B_{21}\right)+8B_{01}\}+O(\epsilon^2), \\ \dot{\bar{\phi}} &= -\frac{\epsilon\bar{r}^2}{16}\left(B_{32}\bar{r}^2\omega^2+2B_{12}\omega^2+6B_{30}\right)+O(\epsilon^2). \end{aligned} \tag{6.10}$$

A close scrutiny reveals that the effect on $\dot{\bar{r}}$ arises only in terms of the even coefficients of $F(\xi,\dot{\xi})$ in first order approximation. Again, only odd polynomial $G(\xi)$ plays a role in phase equation. One thus finds at most four non-zero values of $\bar{r}$ having three distinct possibilities: (a) two sets of complex conjugate roots with asymptotically stable solution, (b) one pair of complex conjugate roots along with two real roots of equal magnitude having opposite sign giving a limit cycle solution and (c) either four real roots of equal magnitude of double multiplicity with opposite sign gives a limit cycle solution or two different sets of real roots of equal magnitude having opposite sign will provide limit cycle solutions of different radius.

The roots of $\bar{r}$ with unique zero value gives a center [49, 97]. The stability of the cycles thus can be examined by the $-ve$ or $+ve$ sign of $\frac{d\dot{\bar{r}}}{d\bar{r}}|_{\bar{r}=\bar{r}_{ss}}$ where $\frac{d\dot{\bar{r}}}{d\bar{r}}|_{\bar{r}=\bar{r}_{ss}=0}>0$ or <0 which determines the nature of the fixed point. Note that the limit cycle condition, $F(0,0)<0$ fails to give any clue about the number of limit cycles a system can admit. According to the root finding algorithm one can guess the maximum number of limit cycles of a LLS system[96, 255, 256].

On a more general footing we have three possible combinations for having the maximum number of limit cycles or distinct non-zero real roots, which are given below:

Case-I : For both even or odd N and M there exist maximum $\frac{N+M}{2} - 1$ limit cycles.

Case-II : For even N and odd M there exist maximum $\frac{N+M-1}{2}$ limit cycles.

Case-III : For odd N and even M there exist maximum $\frac{N+M-3}{2}$ limit cycles.

Determination of maximum number of possible limit cycles does not necessarily ensure the explicit functional form of the polynomials for damping and restoring forces, since the magnitude of the coefficients remain unknown. An important step in this direction is the systematic estimation of the parameter values or the coefficients of the polynomials for practical realisation of the oscillator with multiple limit cycles. In the next two sections we proceed to deal with this issue.

6.3 On the generalisation of single-cycle oscillator to multicycle cases

We now introduce the models for multicycle cases for Van der Pol and Rayleigh family of oscillators for k-cycles. In view of the above approach, we begin examining some physical models by classifying them according to single, two, three and k cycles for both families of oscillators which are available in the literature. The connection is discussed for the general model system with reference to the polynomial form of damping and restoring forces.

6.3.1 *Van der Pol family of cycles*

For Van der Pol family of oscillators with k-cycles[39, 270, 272] the equation in the LLS form is

$$\ddot{x} + (a_1 + a_2 x + \cdots + a_{2k+1} x^{2k})\dot{x} + x = 0. \tag{6.11}$$

Considering the special case of Van der Pol oscillator developed by equation, $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$ for weak nonlinearity with $0 < \epsilon \ll 1$, we obtain a stable limit cycle with $F(0,0) < 0$. By referring to the earlier case-II as considered in the last section we have $N = 2$ and $M = 1$, which gives a unique stable limit cycle.

Two cycle cases are relatively rare. We rewrite the Liénard equation in the standard form as, $x\dot{(t)} = y(t) - F(x(t)), y\dot{(t)} = -x(t)$ with $F(x(t))$ being an odd polynomial. Subsequently we have, $\ddot{x} + F'(x)\dot{x} + x = 0$ with $F'(x) = \frac{\partial F(x)}{\partial x}$ which now becomes an even polynomial. For $F(x) = a_1 x + a_2 x^2 + a_3 x^3$, it has been shown[39, 266, 270, 272] that the system has a unique limit cycle if $a_1 a_3 < 0$, which will be stable if $a_1 < 0$ and unstable if $a_1 > 0$. It corresponds to case-II for $N = 2, M = 1$. It is further extended by Rychkov[271] and showed that for $F(x) = (a_1 x + a_3 x^3 + a_5 x^5)$ the number of limit cycles is at most two. This observation corroborates with the numerical simulation when $F(x)$ is chosen[39, 266, 270, 272] as, $F(x) = 0.32x^5 - \frac{4}{3}x^3 + 0.8x$. Here the inner limit cycle is an unstable one as $F(0,0) = 0.8 > 0$ but the outer one is stable. This correspond to case-II with $N = 4$ and $M = 1$.

Extended Van der Pol oscillator for three cycle case is coined by Kaiser[55, 56, 85, 89, 90, 267] which is described as bi-rhythmicity, having the form,

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + x = 0. \tag{6.12}$$

Here the parameters $\alpha, \beta, \mu > 0$ of the model provides an extremely rich variety of bifurcation phenomena and it exhibits bi-rhythmicity. The model allows three limit cycles with two stable cycle and in between the two stable limit cycles there will be an unstable one dividing the basins of attraction. In presence of an external electric field for the driven Kaiser model, the system shows some interesting phenomena[55, 56, 85, 89, 90, 267]. This model corresponds to the case-II with $N = 6$ and $M = 1$. Therefore six roots may arise from the amplitude equation for $\mu > 0$ and the controlling parameters, (α, β) have to be chosen from the zone of three limit cycles. Furthermore for $\beta = 0$, the undriven Kaiser model with $\alpha = 0.1$ gives two limit cycles of which the smaller one will be stable and the larger one will be unstable. For LLS system, Blows and Lloyd[39, 272] have showed that, $\dot{x} = y - F(x), \dot{y} = -g(x)$ with $g(x) = x$ and $F(x) = a_1 x + a_2 x^2 + \cdots + a_{2k+1} x^{2k+1}$ has at most k limit cycles for the cases where the coefficients, $a_1, a_3, \ldots, a_{2k+1}$ alternates in sign. This corresponds to the case for $N = 2k$ and $M = 1$ as the condition of at most k limit cycles.

6.3.2 *Rayleigh family of cycles*

For Rayleigh family of oscillators with k-cycles[264] the equation may be written in LLS form as ,

$$\ddot{x} - \left(\mu_1 + \mu_2 \dot{x} + \mu_3 \dot{x}^2 + \cdots + \mu_{2k} \dot{x}^{2k-1} + \mu_{2k+1} \dot{x}^{2k}\right) \dot{x} + x \quad = 0, \tag{6.13}$$

Furthermore, considering the special case of ordinary Rayleigh oscillator[9, 16],

$$\ddot{x} + \left(\eta_1 \dot{x}^2 - \eta_2\right) \dot{x} + \omega^2 x \quad = 0, \quad \eta_1, \eta_2, \omega > 0,$$

one obtains an unique stable limit cycle for $\eta_2 > 0$. It corresponds to the case-I with $N = 1$ and $M = 3$.

Multiple limit cycles[39, 264, 266] *in Rayleigh family are not known in literature*. Gaiko[264] has shown through geometrical approach that for a Liénard-type system having the form of Eq. (6.13), one can have at most k limit cycles iff $\mu_1 > 0$ and *no physical example is available*. This result[264] is nicely matched[96] and for any value of $k \in \mathbb{Z}$ it corresponds to the odd-odd subcase (see case-I) with $M = 2k + 1$ and $N = 1$ which subsequently gives the at most $\frac{N+M}{2} - 1 = k$ number of cycles.

6.4 Construction of new families of Van der Pol and Rayleigh oscillators with multiple limit cycles

Having discussed the bi-rhythmic Van der Pol oscillator, i.e, the Kaiser model, we now explore tri-rhythmic cases as further generalizations. We begin with alternative generalisation of Van der Pol system for bi-rhythmic and tri-rhythmic cases with higher powers of velocity variables. Similarly, bi-rhythmic and tri-rhythmic models are worked out for the family of Rayleigh oscillator.

6.4.1 *Van der Pol family of oscillators*

6.4.1.1 *Generalisation of Van der Pol system; three stable limit cycles*

We return to LLS oscillator and extend the Kaiser model by keeping a proper combination of N and M for case-II and an alternate sign condition on the coefficients of polynomial[39, 272]. An extended model beyond Kaiser that includes one more stable limit cycle can be written as,

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6 + \gamma x^8 - \delta x^{10})\dot{x} + x = 0, \quad 0 < \mu \ll 1. \tag{6.14}$$

Here μ is the Hopf bifurcation parameter as in Van der Pol model with the system parameters, $\alpha, \beta, \gamma, \delta > 0$. According to case-II the maximum number of limit cycles the system can give is $\frac{10+1-1}{2} = 5$. Further extension of Kaiser model leaving the parameter space (α, β)

remain unchanged, we find the parameter space for (γ, δ) which admits of three stable limit cycles. The associated amplitude equation for Eq. (6.14), is given by,

$$\dot{\bar{r}} \;=\; \frac{\bar{r}\mu}{1024}\left(-21\delta\bar{r}^{10} + 28\gamma\bar{r}^{8} - 40\beta\bar{r}^{6} + 64\alpha\bar{r}^{4} - 128\bar{r}^{2} + 512\right). \tag{6.15}$$

For $\gamma = \delta = 0$ amplitude equation reduces to that for Kaiser model. The corresponding parameter space for (α, β) and phase portrait of bi-rhythmicity (having amplitudes $\bar{r}_{ss} = 2.63902,\ 3.96164$ and 4.83953) of Kaiser model are shown in Fig. 6.1.

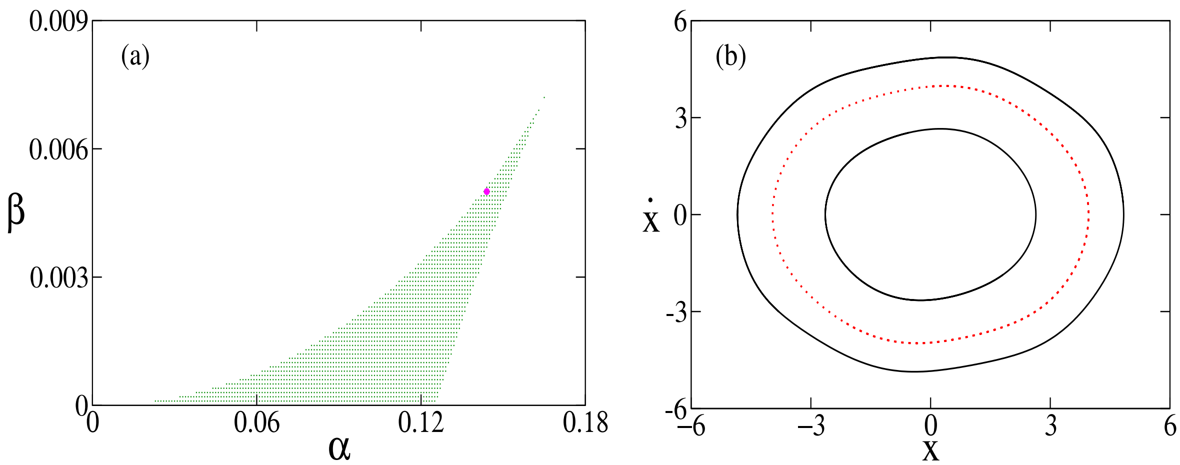

Figure 6.1: *Bi-rhythmic Van der Pol or Kaiser model* (Eq. 6.14 ; $\gamma = \delta = 0$ with $\mu = 0.1$). Subplot (a) represents the bi-rhythmic parameter space for (α, β) and (b) refer to the corresponding phase space plot showing the location of the stable limit cycles (black, continuous) along with an unstable limit cycle (red, dotted) for the parameters values, $\alpha = 0.144$ and $\beta = 0.005$ (magenta dot in subplot a).

To realize tri-rhythmicity in Van der Pol system we search for the specific region of (γ, δ) for a chosen set of values of (α, β) corresponding to the bi-rhythmic space at $(\alpha = 0.144, \beta = 0.005)$, respectively. By solving Eq. (6.15) for the five distinct real roots, one obtains the region as shown in Fig. 6.2(a). As no direct simple method is available for an equation with degree higher than 3, we take resort to numerical simulation using higher order root finding algorithm in Mathematica as the amplitude equation is a polynomial in $\bar{r}$ of degree 11 which can be reduced to degree 6, where one root is zero and other non-zero roots appear in conjugate pairs.

Now, for γ and δ from the region in Fig. 6.2(a), we obtain five distinct magnitudes of non-zero real roots of Eq. (6.15). The number of limit cycles can be obtained from the magnitude of the radii. The unique zero value as a root provides the location of the fixed point as well as the stability of the fixed point of the system. For $\gamma = 0.00005862$ and $\delta = 2.13 \times 10^{-7}$, it gives $\bar{r}_{ss} = 0,\ 2.66673,\ 3.53498,\ 8.3682,\ 10.4793,\ 12.9421$. The stability of the corresponding

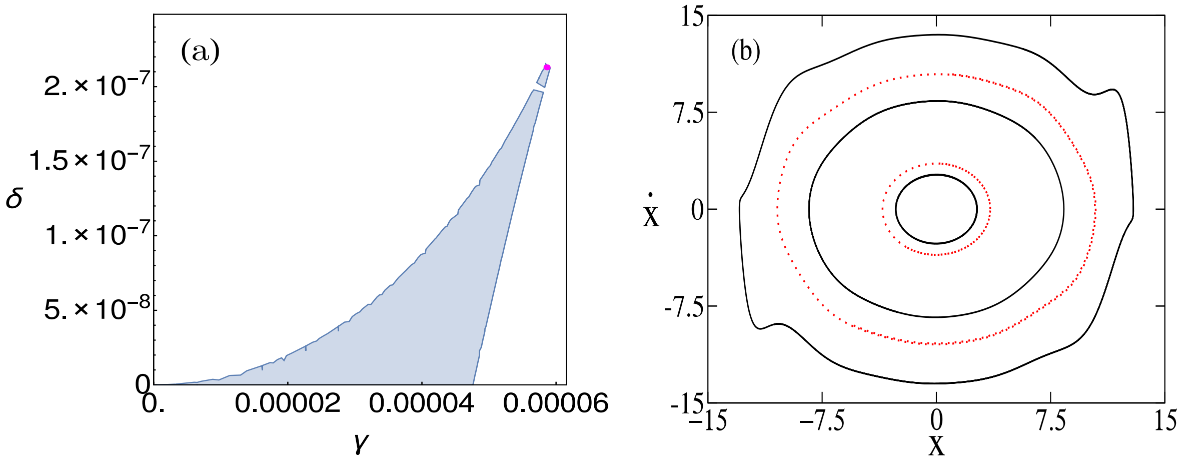

Figure 6.2: *Tri-rhythmic Van der Pol model* (Eq. 6.14 ; $\mu = 0.01$). Subplot (a) represents the tri-rhythmic parameter space of (γ, δ) and (b) refer to the corresponding phase space plot of five concentric limit cycles among them three are stable (black, continuous) and remaining two are unstable (red, dotted) for the parameter values $\alpha = 0.144$, $\beta = 0.005$, $\gamma = 0.00005862$ and $\delta = 2.13 \times 10^{-7}$ (magenta dots in subplot a and Fig. 6.1a).

cycles can be determined by the signs of the real parts of the eigenvalues appearing in the respective order as $(-, +, -, +, -)$ for $\bar{r}_{ss} = 2.66673,\ 3.53498,\ 8.3682,\ 10.4793,\ 12.9421$. This implies that the cycles are in the order of stable, unstable, stable, unstable, stable, respectively. The unique fixed point $\bar{r}_{ss} = 0$ (i.e., the origin) is unstable as $\frac{d\dot{\bar{r}}}{d\bar{r}}|_{\bar{r}_{ss}=0} > 0$. The phase space plot of Fig. 6.2(b) corresponds to the stable cycles (black, continuous) along with unstable cycles (red, doted). One has to set μ is very small[39, 272] as much as possible to get nearly a circular orbit.

6.4.1.2 *Another generalization of Van der Pol system: three stable limit cycles*

We now examine the LLS oscillator model to suggest an alternative extension of Van der Pol model, where the damping part contains $\dot{x}^3$ instead of $\dot{x}$ as follows,

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6 + \gamma x^8 - \delta x^{10})\dot{x}^3 + x = 0,\ 0 < \mu \ll 1;\ \alpha, \beta, \gamma, \delta > 0. \tag{6.16}$$

This corresponds to case-II, $N = 10$ and $M = 3$; therefore the system can have atmost $\frac{10+3-1}{2} = 6$ limit cycles. The numerical examination however, reveals that the actual number is 5. The justification is provided below. The corresponding amplitude equation takes the form,

$$\dot{\bar{r}} \ = \frac{\bar{r}^3\mu}{2048}\left(-9\delta\bar{r}^{10} + 14\gamma\bar{r}^8 - 24\beta\bar{r}^6 + 48\alpha\bar{r}^4 - 128\bar{r}^2 + 768\right). \tag{6.17}$$

Now, to have a complete knowledge of the full parameter space for tri-rhythmicity for $(\alpha, \beta, \gamma, \delta)$–at first we have to fix $\gamma = \delta = 0$, as in the previous case. This is a generically distinct kind of variant of Kaiser model having three limit cycles. This type of bi-rhythmic oscillator is not known in the literature. For $\gamma = \delta = 0$, case-II gives the maximum number of limit cycles 4, but amplitude equation (6.17) says that it has at most three non-zero roots of distinct magnitudes and the zero root has multiplicity three. The bi-rhythmic parameter zone and the phase space plot for $\gamma = \delta = 0$ are given in Fig. 6.3.

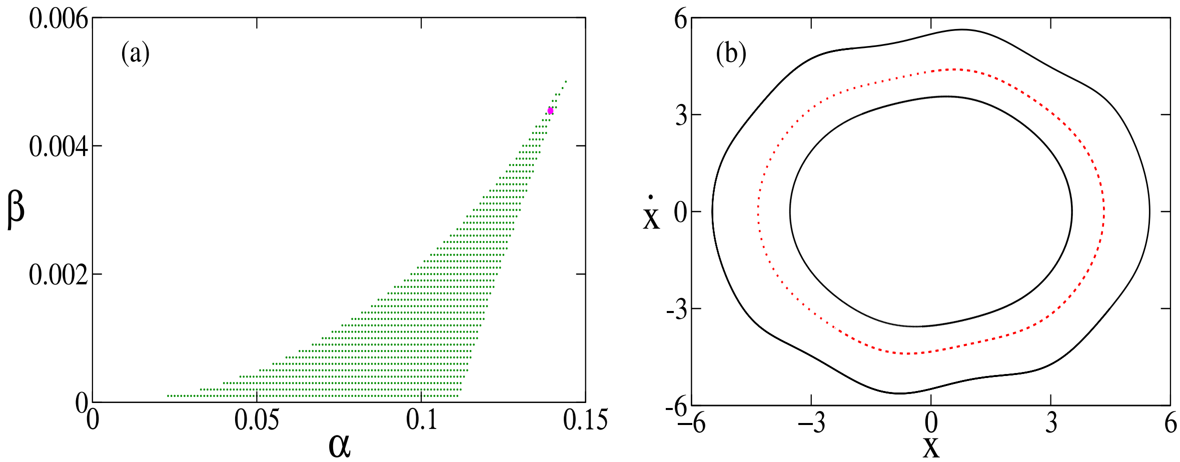

Figure 6.3: *A generalisation of bi-rhythmic Van der Pol oscillator* (Eq. 6.16 ; $\gamma = 0 = \delta$ with $\mu = 0.01$). Subplot (a) represents the bi-rhythmic parameter space for (α, β) and (b) refer to the corresponding phase space plot showing the location of the stable limit cycles (black, continuous) along with the unstable limit cycle (red, dotted) for the parameters values, $\alpha = 0.139317$ and $\beta = 0.00454603$ (magenta dot in subplot a).

Now, for $\alpha = 0.139317$ and $\beta = 0.00454603$, one must have three non-zero values of $\bar{r}_{ss} = 3.53297,\ 4.33345,\ 5.48006$. The zero value of $\bar{r}_{ss}$ has multiplicity 3 which is basically a neutral fixed point but is unstable in nature. The stability of the cycles is given in an outward sequence as stable, unstable and stable, respectively.

Now, to find the region of (γ, δ) in 2-D space, we first fix the parameters at $\alpha = 0.139317$ and $\beta = 0.00454603$. Eq. (6.17) is then solved to provide five distinct real roots. The parameter region for (γ, δ) is given in Fig. 6.4(a). For fixed values of $\gamma = 0.00002402$ and $\delta = 3.058 \times 10^{-8}$– the distinct roots of $\bar{r}_{ss}$ are 0(of multiplicity 3), $3.75166,\ 3.80059,\ 6.63736,\ 20.9299,\ 26.6688$ and the non-zero values are the radii of the cycles having the stability in outward sequence as: stable, unstable, stable, unstable, stable, respectively. The respective phase space portrait is given in Fig. 6.4(b).

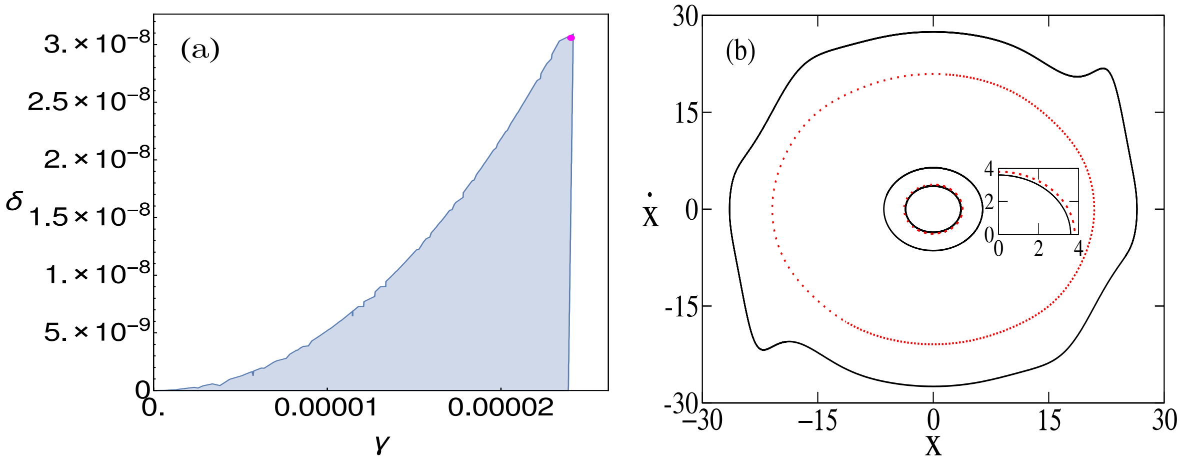

Figure 6.4: *Alternate generalisation of tri-rhythmic Van der Pol oscillator* (Eq. 6.16 ; $\mu = 0.0000001$). Subplot (a) represents the tri-rhythmic parameter space of (γ, δ) and (b) refer to the corresponding phase space plot of five concentric limit cycles among them three are stable (black, continuous) and two are unstable (red, dotted) for the parameter values $\alpha = 0.139317$, $\beta = 0.00454603$, $\gamma = 0.00002402$ and $\delta = 3.058 \times 10^{-8}$ (magenta dots in subplot a and Fig. 6.3a). The inset in subplot (b) zooms the gap between the first stable and unstable limit cycles in the outward direction.

6.4.2 *Rayleigh family of oscillators*

6.4.2.1 *Bi-rhythmic Rayleigh: three stable limit cycles*

Here we consider an extension for Rayleigh oscillator model as a LLS system. The model can be written as,

$$\ddot{x} - \mu(1 - \dot{x}^2 + \alpha\dot{x}^4 - \beta\dot{x}^6 + \gamma\dot{x}^8 - \delta\dot{x}^{10})\dot{x} + x = 0, \; 0 < \mu \ll 1; \; \alpha, \beta, \gamma, \delta > 0. \tag{6.18}$$

Corresponding to case-I, we have $N = 1$ and $M = 11$ and accordingly the system can have at most $\frac{11+1-2}{2} = 5$ limit cycles. The amplitude equation takes the form,

$$\dot{\bar{r}} \; = \frac{\bar{r}\mu}{1024}\left(-231\delta\bar{r}^{10} + 252\gamma\bar{r}^8 - 280\beta\bar{r}^6 + 320\alpha\bar{r}^4 - 384\bar{r}^2 + 512\right). \tag{6.19}$$

Proceeding as in the previous case we see that for $\gamma = \delta = 0$ the maximum number of limit cycles is 3 having the bi-rhythmic parameter region (α, β) as given in Fig. 6.5(a). The corresponding phase portrait is shown in Fig. 6.5(b) with amplitudes $\bar{r}_{ss} = 1.69091, \; 2.03334$ and 2.51274.

For searching the region of (γ, δ) in 2-D space, (α, β) is fixed at $(0.285272, 0.0244993)$. The parameter space (γ, δ) is shown in Fig. 6.6(a). The phase portrait in Fig. 6.6(b) shows the tri-rhythmicity for $\gamma = 0.0002544$ and $\delta = 6.62 \times 10^{-7}$. The amplitudes for the tri-rhythmic

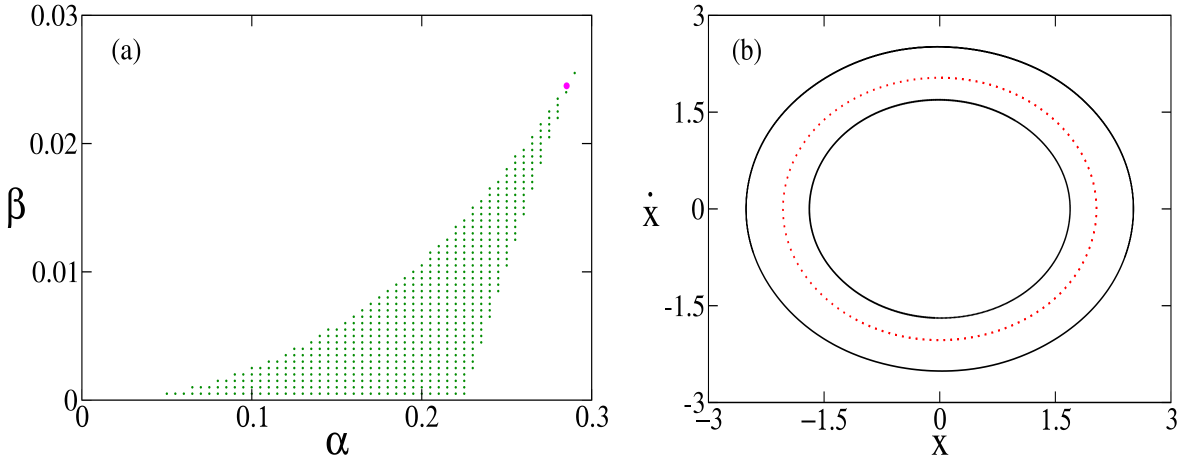

Figure 6.5: *Bi-rhythmic Rayleigh oscillator* (Eq. 6.18 ; $\gamma = 0 = \delta$ with $\mu = 0.1$). Subplot (a) represents the bi-rhythmic parameter space for (α, β) and (b) refer to the corresponding phase space plot showing the location of the stable limit cycles (black, continuous) along with an unstable limit cycle (red, dotted) for the parameters values, $\alpha = 0.285272$ and $\beta = 0.0244993$ (magenta dot in subplot a).

Rayleigh oscillator, are , $\bar{r}_{ss} =$ 1.77779, 1.82091, 2.86779, 12.5239 and 15.7377–are in the same outward sequence i.e. stable, unstable, stable, unstable and stable.

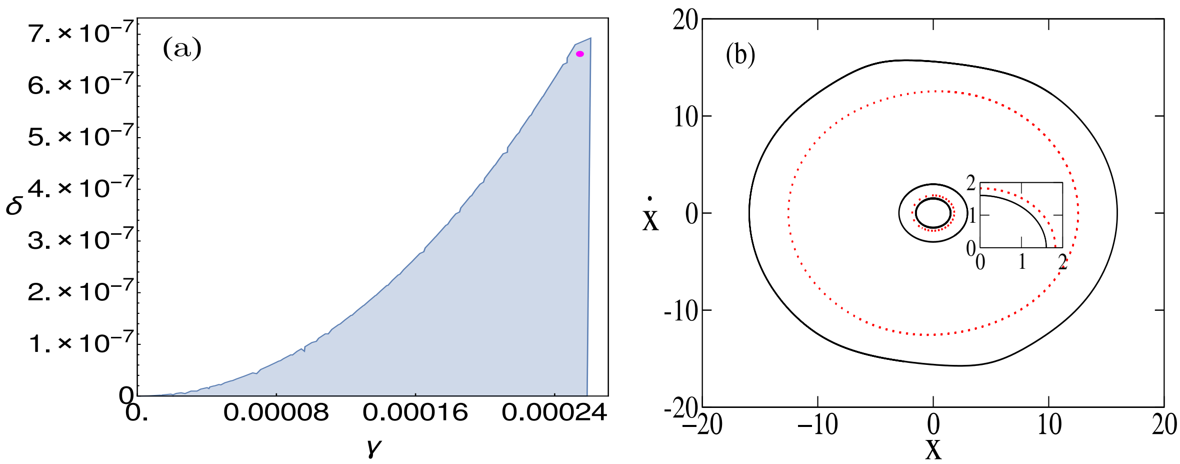

Figure 6.6: *Tri-rhythmic Rayleigh oscillator* (Eq. 6.18 ; $\mu = 0.00001$). Subplot (a) represents the tri-rhythmic parameter space of (γ, δ) and (b) refer the corresponding phase space plot of five concentric limit cycles among them three are stable (black, continuous) and two are unstable (red, dotted) for the parameter values $\alpha = 0.285272$, $\beta = 0.0244993$, $\gamma = 0.0002544$ and $\delta = 6.62 \times 10^{-7}$ (magenta dots in subplot a and Fig. 6.5a). The inset in subplot (b) zooms the gap between the first stable and unstable limit cycles in the outward direction.

6.4.2.2 *Alternative form of extended Rayleigh model: three stable limit cycles*

Here we consider another LLS oscillator system as an alternative form of the extended Rayleigh oscillator model with $N = 1$ and $M = 13$. The model can be written as,

$$\ddot{x} - \mu(1 - \dot{x}^2 + \alpha\dot{x}^4 - \beta\dot{x}^6 + \gamma\dot{x}^8 - \delta\dot{x}^{10})\dot{x}^3 + x = 0,\ 0 < \mu \ll 1;\ \alpha, \beta, \gamma, \delta > 0. \qquad (6.20)$$

As per case-I, the system has at most $\frac{13+1-2}{2} = 6$ cycles and the corresponding amplitude equation takes the form,

$$\dot{\bar{r}} \ = \frac{\mu\bar{r}^3}{2048}\left(-429\delta\bar{r}^{10} + 462\gamma\bar{r}^8 - 504\beta\bar{r}^6 + 560\alpha\bar{r}^4 - 640\bar{r}^2 + 768\right). \qquad (6.21)$$

For $\gamma = \delta = 0$ we have at most 3 limit cycles having the bi-rhythmic parameter region for (α, β) as shown in Fig. 6.7(a). The corresponding phase portrait is given in Fig. 6.7(b). Here, for the above bi-rhythmic Rayleigh oscillator the amplitudes will take the values, $\bar{r}_{ss} = 1.5775,\ 1.90947$ and 2.52202.

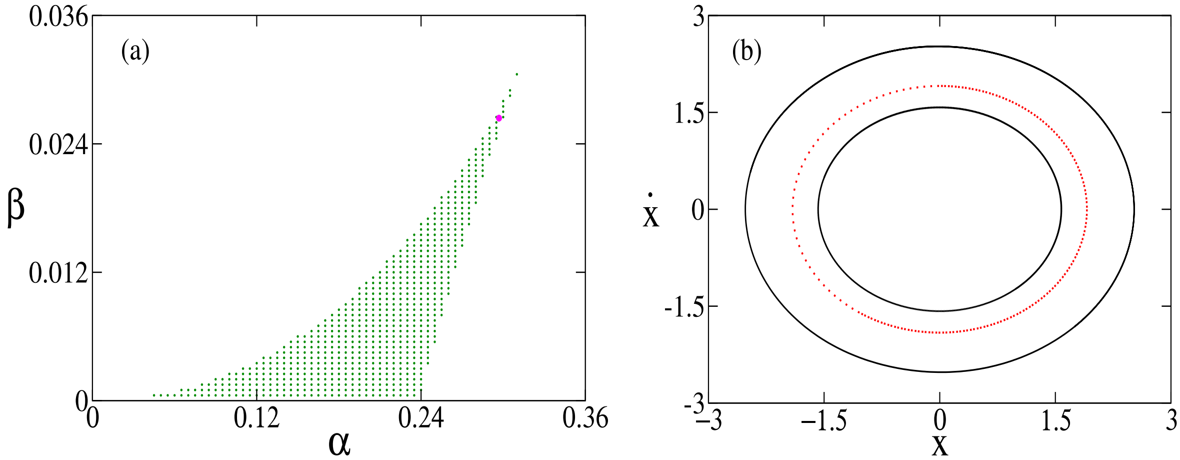

Figure 6.7: *Alternate generalisation of Rayleigh oscillator for Bi-rhythmicity* (Eq. 6.20 ; $\gamma = 0 = \delta$ with $\mu = 0.01$). Subplot (a) represents the bi-rhythmic parameter space for (α, β) and (b) refer to the corresponding phase space plot showing the location of the stable limit cycles (black, continuous) along with an unstable limit cycle (red, dotted) for the parameters values, $\alpha = 0.296930$ and $\beta = 0.0264040$ (magenta dot in subplot a).

The parameter space of (γ, δ) in 2-D space for a fixed $\alpha = 0.296930$ and $\beta = 0.0264040$, is shown in Fig. 6.8(a) and Fig. 6.8(b) shows the corresponding tri-rhythmic phase portrait at a fixed values $(\gamma,\ \delta)$ i.e., $\gamma = 0.0004334$ and $\delta = 1.815 \times 10^{-6}$. The amplitudes of Eq. (6.21) will be the non-zero steady states values of $\bar{r}_{ss}$ i.e., $1.66034,\ 1.70743,\ 3.0214,\ 9.27171$ and 12.5056, which are in the same stability sequence as in previous.

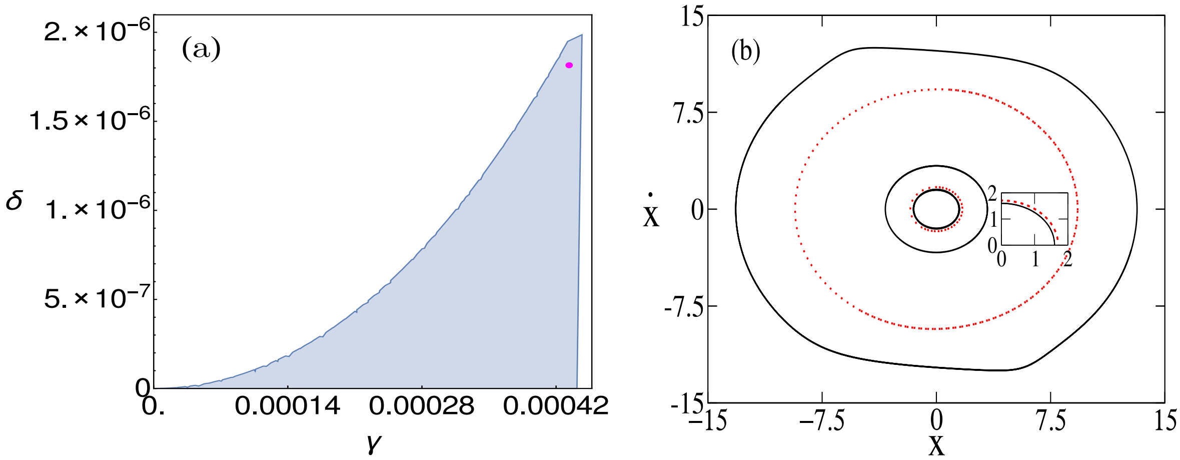

Figure 6.8: *Alternate generalisation of Rayleigh oscillator for tri-rhythmicity* (Eq. 6.20 ; $\mu = 0.0000005$). Subplot (a) represents the tri-rhythmic parameter space of (γ, δ) and (b) refer to the corresponding phase space plot of five concentric limit cycles among them three are stable (black, continuous) and two are unstable (red, dotted) for the parameter values $\alpha = 0.296930$, $\beta = 0.0264040$, $\gamma = 0.0004334$ and $\delta = 1.815 \times 10^{-6}$ (magenta dots in subplot a and Fig. 6.7a). The inset in subplot (b) zooms the gap between the first stable and unstable limit cycles in the outward direction.

By making use of this approach we have demonstrated the parameter spaces for bi-rhythmicity and tri-rhythmicity for Rayleigh and Van der Pol cases. The structure of the stable and unstable concentric multiple limit cycles are described in terms of corresponding radial equations. The scheme also covers the alternate generalizations of bi-rhythmic and tri-rhythmic Rayleigh and Van der Pol families of oscillators.

For the alternate cases (i.e., Eq. 6.16 and Eq. 6.20), both the families of oscillators it appears that $F(0,0) = 0$ which is a condition for center[97]. However, both of the families have at least one stable limit cycle which can be checked from the amplitude equation, giving a non-zero radius. It can be resolved if we consider a small neighbourhood of $(0,0)$, say, (δ_1, δ_2) where $F(\delta_1, \delta_2)_{(\delta_1,\delta_2)\to(0,0)} < 0$.

6.5 BI-RHYTHMICITY IN OTHER SYSTEMS

In the above main chapter we have discussed how a class of mono-rhythmic system expressed as LLS system can be made bi-rhythmic. LLS system is of a a typical kinetic form, say, $\dot{x} = y$, $\dot{y} = -g(x) - f(x, \dot{x})\dot{x}$, but there might have some more general form other than this standard one. So, a question may arise, once we have a bi-rhythmic system in a LLS form then what will be its non-trivial kinetic set of equations? We have raised this point in Appendix C. In Appendix C.1 we have discussed how to generate a kinetic form of a bi-rhythmic oscillator starting from a kinetic ordinary differential equation (ODE) by choosing Schnakenberg model[40] through casting them into a LLS form along with the addition of higher order nonlinearity (Liénard-type extension). The Rayleigh type extension is also discussed in Appendix C.1.1. Finally, a bi-rhythmic $\lambda - \omega$ system is discussed in Appendix C.2 through a proper example.

6.6 SUMMARY, DISCUSSIONS AND CONCLUSIONS

Based on a general scheme of counting limit cycles of a given LLS equation we have proposed a recipe for systematically designing models of multi-rhythmicity. We note the basic tenets of the scheme stepwise as follows;

Step-I: Given that the number of desired limit cycles is k, we may partition k into N and M such that $k = (\frac{N+M}{2} - 1)$ for Case-I; $k = (\frac{N+M-1}{2})$ for Case-II and $k = (\frac{N+M-3}{2})$ for Case-III taking care of even and odd nature of N and M as appropriate for Van der Pol or Rayleigh families of oscillators, M and N being the highest power of velocity $\dot{\xi}$ and position ξ, respectively for the damping function $F(\xi, \dot{\xi})$ and force function $G(\xi)$.

Step-II: Once N and M are fixed we need to construct the polynomials for damping functions taking care of alternative signs and polynomial for the force function.

Step-III: To fix the coefficients of the polynomials we first consider the lowest order polynomial functions that determine the mono or bi-rhythmic model. By choosing a point in the two-dimensional parameter space for the specific non-zero values of the coefficients for the models with high multi-rhythmicity are switched on. For example, as in section 6.4.1, we choose one point from $\alpha - \beta$ parameter space for bi-rhythmic model and then proceed to construct $\gamma - \delta$ space (for a fixed $\alpha,\ \beta$) for the model of tri-rhythmicity. The procedure may be repeated for higher order variants.

Step-IV: A smallness parameter (μ as in Eq. 6.14) can be introduced in the damping term which can be suitably tuned for numerical realisation of the high order limit cycles.

In Table 6.1 we have summarised several cases of oscillators belonging to Van der Pol and Rayleigh families of oscillators. In terms of the polynomial form of the damping and forcing functions of phase space variables the average amplitude and phase equations can be derived to obtain the maximum possible number of limit cycles of a dynamical system. It is verified that for a LLS system depending on the values of N and M which are the maximum degree of the polynomials for damping and restoring forces, respectively, the number of limit cycles can be at most $\frac{N+M-2}{2}$ (even-even or odd-odd sub cases) or $\frac{N+M-1}{2}$ (even-odd sub case) or $\frac{N+M-3}{2}$ (odd-even sub case). The generalized Liénard system can be recovered for $M = 1$ and $N \in \mathbb{Z}^+$ whereas for the generalised Rayleigh oscillator we have, $N = 1$ and $M \in \mathbb{Z}^+$. For polynomial form of damping and restoring force functions we have constructed bi-rhythmic and tri-rhythmic oscillators of Van der Pol and Rayleigh families. New alternative generalizations of Van der Pol and Rayleigh families of oscillators are also introduced as models for bi-rhythmicity and tri-rhythmicity. Our approach shows that it is possible to construct a LLS system with arbitrary rhythmicity.

Secondly the scheme is used to determine the appropriate range of parameters for realizing limit cycle oscillations as shown by the corresponding phase portraits. Our approach shows that once the parameters space that allows a single limit cycle is determined, one may choose a suitable point in this space to select only the parameters pertaining to the higher order terms of the polynomials for a systematic search for parameter space for constructing the remaining limit cycles. Since the choice of the point as referred to is not unique, it is imperative that the choice of parameters remains widely open for various generalisation of a multi-rhythmic system.

As multi-rhythmicity plays an important role in switching transitions between different dynamical states in a nonlinear system, its control[273] and manipulation[89, 274] would be useful in many self-induced oscillatory processes[10, 28, 36] in diverse interdisciplinary areas. The proposed multi-rhythmic oscillators can also be useful in various circuit designs as well as networks according to the demand of the system upon using different controlling schemes (e.g., delay-feedback control[56], self-conjugate feedback control[91, 92] and a number of approaches reviewed by Pisarchik et al.[275]) to convert them into lower rhythmic systems. Systematic lowering of rhythmicity may also be utilized by suitably reducing the reaction rates of higher order polymeric reactions, using such schemes of control.

$h(x,\ \dot{x})$	$f(r)$	Stability of FP	Regions & fixed $(\alpha,\ \beta)$	Stability & radii	
$-(1-x^2+\alpha x^4-\beta x^6)\dot{x}$	$\bar{r}\left(-5\beta\bar{r}^6+8\alpha\bar{r}^4-16\bar{r}^2+64\right)$	U	Fig. 6.1(a) $(0.144,\ 0.005)$	S, U, S 2.63902, 3.96164, 4.83953	Bi -r
$-(1-x^2+\alpha x^4-\beta x^6)\dot{x}^3$	$\bar{r}^3\left(-3\beta\bar{r}^6+6\alpha\bar{r}^4-16\bar{r}^2+96\right)$	NS	Fig. 6.3(a) $(0.139317,\ 0.00454603)$	S, U, S 3.53297, 4.33345, 5.48006	hy th
$-(1-\dot{x}^2+\alpha\dot{x}^4-\beta\dot{x}^6)\dot{x}$	$\bar{r}\left(-35\beta\bar{r}^6+40\alpha\bar{r}^4-48\bar{r}^2+64\right)$	U	Fig. 6.5(a) $(0.285272,\ 0.0244993)$	S, U, S 1.69091, 2.03334, 2.51274	mi c
$-(1-\dot{x}^2+\alpha\dot{x}^4-\beta\dot{x}^6)\dot{x}^3$	$\bar{r}^3\left(70\alpha\bar{r}^4-63\beta\bar{r}^6-80\bar{r}^2+96\right)$	NS	Fig. 6.7(a) $(0.296930,\ 0.0264040)$	S, U, S 1.5775, 1.90947, 2.52202	
$-(1-x^2+\alpha x^4-\beta x^6+\gamma x^8-\delta x^{10})\dot{x}$	$\bar{r}\left(-21\delta\bar{r}^{10}+28\gamma\bar{r}^8-40\beta\bar{r}^6+64\alpha\bar{r}^4-128\bar{r}^2+512\right)$	U	Fig. 6.2(a) $(5.862\times10^{-5},\ 2.13\times10^{-7})$	S, U, S, U, S 2.66673, 3.53498, 8.3682, 10.4793, 12.9421	Tr i-
$-(1-x^2+\alpha x^4-\beta x^6+\gamma x^8-\delta x^{10})\dot{x}^3$	$\bar{r}^3\left(-9\delta\bar{r}^{10}+14\gamma\bar{r}^8-24\beta\bar{r}^6+48\alpha\bar{r}^4-128\bar{r}^2+768\right)$	NS	Fig. 6.4(a) $(2.402\times10^{-5},\ 3.058\times10^{-8})$	S, U, S, U, S 3.75166, 3.80059, 6.63736, 20.9299, 26.6688	rh yt
$-(1-\dot{x}^2+\alpha\dot{x}^4-\beta\dot{x}^6+\gamma\dot{x}^8-\delta\dot{x}^{10})\dot{x}$	$\bar{r}\left(-231\delta\bar{r}^{10}+252\gamma\bar{r}^8-280\beta\bar{r}^6+320\alpha\bar{r}^4-384\bar{r}^2+512\right)$	U	Fig. 6.6(a) $(2.544\times10^{-4},\ 6.62\times10^{-7})$	S, U, S, U, S 1.77779, 1.82091, 2.86779, 12.5239, 15.7377	hm ic
$-(1-\dot{x}^2+\alpha\dot{x}^4-\beta\dot{x}^6+\gamma\dot{x}^8-\delta\dot{x}^{10})\dot{x}^3$	$\bar{r}^3\left(-429\delta\bar{r}^{10}+462\gamma\bar{r}^8-504\beta\bar{r}^6+560\alpha\bar{r}^4-640\bar{r}^2+768\right)$	NS	Fig. 6.8(a) $(4.334\times10^{-4},\ 1.815\times10^{-6})$	S, U, S, U, S 1.66034, 1.70743, 3.0214, 9.27171, 12.5056	

Table 6.1: Nonlinear damping functions, classification of limit cycles of bi-rhythmic and tri-rhythmic cases of Van der Pol and Rayleigh families of oscillators (S=Stable, U=Unstable, NS=Neutrally Stable, FP=Fixed Point).

7

PERIODICALLY MODULATED NONLINEARITY IN LIMIT CYCLE SYSTEMS: EFFECT OF DELAY AND CONTROL OF BI-RHYTHMICITY

7.1 INTRODUCTION

Recently, the effect of periodically modulating the nonlinearity in a limit cycle system, viz., Van der Pol oscillator has been investigated as a **p**arametrically **e**xcited **n**onlinearity in the **V**an der Pol **o**scillator(PENVO) along with the standard phenomenon of resonance, exhibits the phenomenon of antiresonance that is said to have occurred if there is a decrease in the amplitude of the limit cycle at a certain frequency of the parametrical drive. Time delay is known to have significant effect on the attractors of a nonlinear system and can also brings forth new ones. For example, even in a relatively simple system like the Rössler oscillator, time delayed feedback control [276] induces a large variety of regimes, like tori and new chaotic attractors, non-existent in the original system; furthermore, the delay modifies the periods and the stabilities of the limit cycles in the system depending on the strength of the feedback and the magnitude of the delay. As another example, we may point out that the direct delayed optoelectronic feedback can suppress hysteresis and bistability in a directly modulated semiconductor laser [277]. The co-existence of two stable limit cycles with different frequencies in the presence of delayed feedback has been discussed in detail [278] for the Van der Pol oscillator and its variants. Mutlicycle Van der Pol oscillator has also been investigated from the point of view of control of bi-rhythmicity using some different forms of time delay [56, 89, 91, 92, 267, 268, 274].

However, to the best of our knowledge, there has been no investigation into the control of multi-stability in a parametric oscillator whose parameter, determining the strength of the nonlinear term, is varied. It should be noted that periodic variation of such a parameter is not inconceivable [233]; in fact, it can result in parametric spatiotemporal instability leading to interesting time-periodic stationary patterns in reaction-diffusion systems. In view of the above, it is imperative that an investigation of the PENVO and its relevant extension be

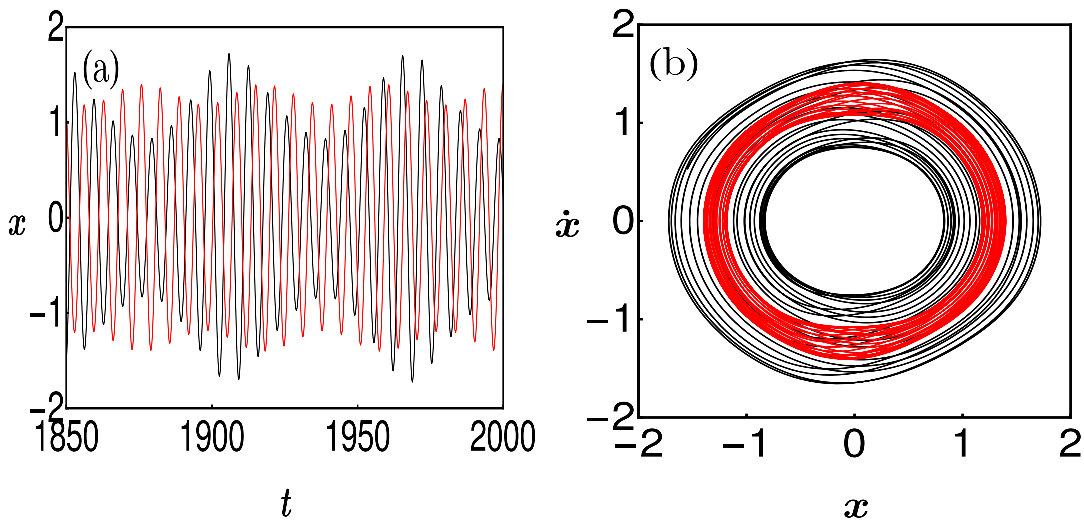

Figure 7.1: *Limit cycles in PENVO with delay have oscillating amplitudes.* We time-evolve Eq. (7.1) with $\gamma = 1.5$, $K = \mu = 0.1$, $\tau = 0.623$ for $\Omega = 2$ (black) and 4 (red) to arrive at the corresponding time-series plots (subplot a), x vs. t, and phase space plots (subplot b), $\dot{x}$ vs. x.

carried out and the interplay, if any, between the time-delayed feedback and the parametric forcing be revealed.

To this end, here[1], we first discuss in section 7.2 how presence of time delayed feedback affects the resonance and the antiresonance in the PENVO. Furthermore, we discuss how the resulting bi-rhythmicity therein is supressed by tuning the strength of the period modulation. Subsequently, in section 7.3, we consider multicycle PENVO—multicycle Van der Pol oscillator whose nonlinearity is sinusoidally varying—and argue in detail that it is possible to control bi-rhythmicity in this system as well. Finally, we reiterate the main results of this chapter in section 7.4.

7.2 PENVO WITH DELAY

Even a simple harmonic oscillator with its quadratic potential modified so as to have a term that is time delayed, exhibits non-trivial dynamics. The resulting solutions, including the oscillatory ones, in the weak nonlinear limit can be iteratively extracted using perturbative methods based on the concept of Renormalisation group [49, 261]. An extended version of the delayed simple harmonic oscillator, that possesses limit cycle, has also been analyzed [97] using the Krylov–Bogolyubov (K-B) method [265, 279]. Motivated by these results, we now consider the PENVO with a time delay term as follows:

$$\ddot{x} + \mu[1 + \gamma\cos(\Omega t)](x^2 - 1)\dot{x} + x - Kx(t - \tau) = 0, \tag{7.1}$$

1 Some portion of this chapter is submitted-Saha et al.(submitted) [arXiv:2007.14883]

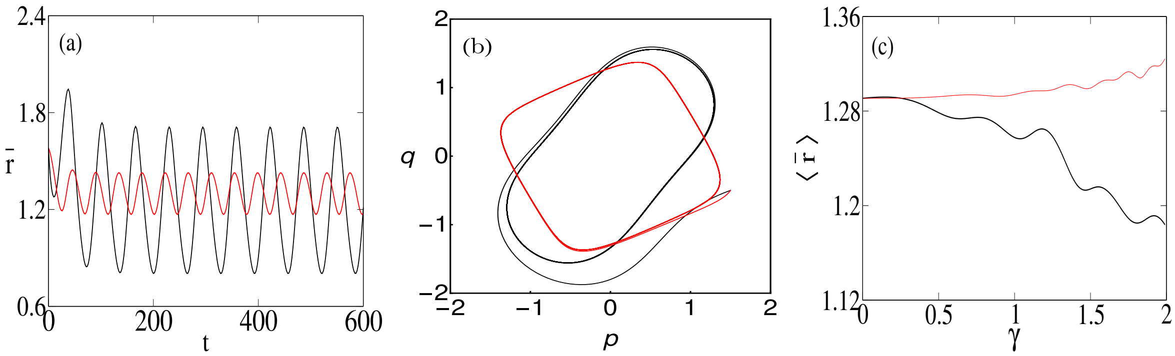

Figure 7.2: *Antiresonant responses with oscillating amplitudes in PENVO with delay.* This figure panel has been generated by time-evolving Eq. (7.1) with $\gamma \in [0,2]$, $K = \mu = 0.1$, $\tau = 0.623$; and $\Omega = 2$ (black) and 4 (red). The time-series, $\bar{r}$ vs. t, (subplot a) depicts oscillating limit cycles in the PENVO with delay and the reason behind the oscillations is best understood as the corresponding non-circular limit cycle attractors in the p-q plane (subplot b). While for subplots (a) and (b), $\gamma = 1.5$, subplot (c) showcases the variation of the averaged amplitudes with γ, thus, highlighting the presence of antiresonances $\forall \gamma \in [0,2]$.

where $0 < K, \mu, \tau \ll 1$; $\gamma \in \mathbb{R}$; and $\Omega \in \mathbb{R}^+$.

Note that for $K = \gamma = 0$, we get back the Van der Pol oscillator that in weak nonlinear limit shows stable limit cycle oscillations with amplitude 2. For appropriate non-zero values of γ (K still zero), we arrive at the equation for the PENVO [93] that is known to show antiresonance (oscillations with amplitude smaller than 2) and resonance (oscillations with amplitude greater than 2) at $\Omega = 2$ and $\Omega = 4$ respectively. *Our specific goal in this section is to find out what happens to the resonance and the antiresonance states once the time delay is introduced (i.e., when $K, \gamma \neq 0$ and $\Omega = 2, 4$), and to explore the possible existence of bi-rhythmicity and its control in the system.*

To begin with we have extensively searched for numerical solutions of Eq. (7.1) at different parameter values. In Fig. 7.1, we present two particular oscillatory solutions for the cases $\Omega = 2$ and $\Omega = 4$. We note that the limit cycles have oscillating amplitudes. In order to understand the origin of oscillating amplitude and to discover bi-rhythimicity in the course of our investigation, we employ the K-B method on Eq. (7.1). We, thus, make an ansatz: $x(t) = r(t)\cos(t + \phi(t))$ where we have adopted polar coordinate, $(r, \phi) = (\sqrt{x^2 + \dot{x}^2}, -t + \tan^{-1}(-\dot{x}/x))$. r and ϕ are very slowly varying function of time since we are working under the assumption that $0 < \mu \ll 1$; we set $r(t) = \bar{r} + O(\mu)$ and $\phi(t) = \bar{\phi} + O(\mu)$. Here, we have used the definition that average of a function, $f(x, \dot{x})$ (say), over a period 2π is conveniently

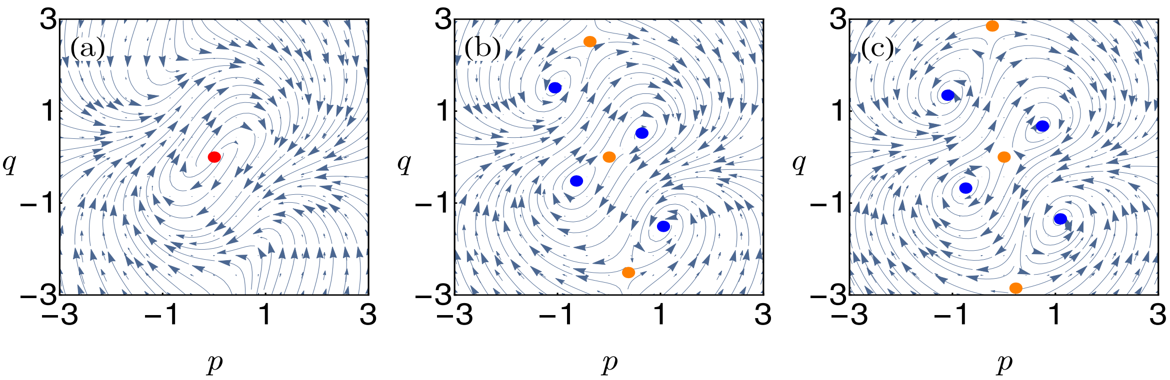

Figure 7.3: *Strength of periodic modulation of nonlinear damping controls delay-induced bi-rhythmicity.* This figure panel of streamline plots depicts repellers [unstable focus (red dot) and saddle (orange dot)] and attractors [stable focus (blue dot) and stable limit cycle (around red dot; not explicitly shown)] in p-q space of the PENVO with delay at $\gamma = 1.5$ (subplot a), 2.5 (subplot b), and 3.3 (subplot c); $K = \mu = 0.1$; $\tau = 0.623$; and $\Omega = 2$. The stable foci on (approximately) principle diagonal of the figures have same $\sqrt{p^2+q^2}$-value, and so is the case with the stable foci on (approximately) anti-diagonal of the figures. Note how with change in γ-value, the number of attractors changes from one (limit cycle) to four (foci that have only two distinct $\sqrt{p^2+q^2}$-value).

denoted as $\overline{f}(t) = (1/2\pi)\int_0^{2\pi} f(s)ds$. Furthermore, Taylor-expanding $r(t-\tau)$ as $r(t-\tau) = r(t) - \tau\dot{r}(t) = r(t) + O(\mu)$ (since $\dot{r}(t) \sim O(\mu)$), one finally obtains

$$\dot{\overline{r}} = -\frac{\overline{r}\left(4K\sin\tau + \mu\left(\overline{r}^2 - 4\right)\right)}{8} + A_\Omega(\overline{r},\overline{\phi};\gamma) + O(\mu^2), \tag{7.2a}$$

$$\dot{\overline{\phi}} = -\frac{K\cos\tau}{2} + B_\Omega(\overline{r},\overline{\phi};\gamma) + O(\mu^2), \tag{7.2b}$$

where, $O(\mu^2)$ terms can be neglected and A_Ω and B_Ω denote the γ dependent parts. It is interesting that these two functions' denominators blow up at Ω equal to 2 and 4. We, thus, resort to the L'Hôspitals' rule to find the functions at $\Omega = 2, 4$:

$$A_2(\overline{r},\overline{\phi};\gamma) = -\frac{1}{4}\gamma\mu\overline{r}\cos\left(2\overline{\phi}\right), \tag{7.3a}$$

$$B_2(\overline{r},\overline{\phi};\gamma) = -\frac{1}{8}\gamma\mu\sin\left(2\overline{\phi}\right)\left(\overline{r}^2 - 2\right); \tag{7.3b}$$

$$A_4(\overline{r},\overline{\phi};\gamma) = \frac{1}{16}\gamma\overline{r}^3\mu\cos\left(4\overline{\phi}\right), \tag{7.3c}$$

$$B_4(\overline{r},\overline{\phi};\gamma) = -\frac{1}{16}\gamma\overline{r}^2\mu\sin\left(4\overline{\phi}\right). \tag{7.3d}$$

Here the subscripts specify the value of Ω at which A_Ω and B_Ω have been determined.

As an illustration, in Fig. 7.2(a), we present $\overline{r}$ as a function of t for both $\Omega = 2$ and $\Omega = 4$ after fixing $\gamma = 1.5$, $\tau = 0.623$, and $K = \mu = 0.1$. The *solutions are oscillatory in sharp contrast* *to the case of the weakly nonlinear Van der Pol oscillator* for which the plot of $\bar{r}$ vs. t would be a horizontal straight line passing through $\bar{r} = 2$ at large times. Obviously, it is a little ambiguous to define the resonance and the antiresonance states in terms of the magnitude of the oscillations' amplitude because the amplitude itself is oscillating. Hence for the sake of consistency, to define the resonance and the antiresonance states, we henceforth use the average of the oscillating amplitude. Consequently, in Fig. 7.2(c), we plot average of $\bar{r}$ i.e. $\langle\bar{r}\rangle_t$ (after removing enough transients) with γ to note that at both $\Omega = 2$ and $\Omega = 4$ the system shows antiresonance. Note that *one of the interesting effects of the delay is to suppress the uncontrolled growth of oscillations* (for $\Omega = 4$ and as $\gamma \to 2$) present in the absence of delay.

The oscillations in the amplitudes of the limit cycles is best explained by recasting the equations for $\bar{r}$ and $\bar{\phi}$ in (p,q)-plane where $(p,q) = \left(\bar{r}\cos\bar{\phi}, \bar{r}\sin\bar{\phi}\right)$ or consequently, $(\bar{r},\bar{\phi}) = (\sqrt{p^2+q^2}, \tan^{-1}(q/p))$. Substituting these relations in equations (7.2), one arrive at the following dynamical flow equations:

$$\dot{p}|_2 = -\frac{Kp\sin\tau}{2} + \frac{Kq\cos\tau}{2} - \frac{\mu p^3}{8} - \frac{\gamma\mu p}{4} + \frac{\mu p}{2} + \frac{1}{4}\gamma\mu p q^2 - \frac{1}{8}\mu p q^2, \tag{7.4a}$$

$$\dot{q}|_2 = -\frac{Kp\cos\tau}{2} - \frac{Kq\sin\tau}{2} - \frac{1}{4}\gamma\mu p^2 q - \frac{1}{8}\mu p^2 q - \frac{\mu q^3}{8} + \frac{\gamma\mu q}{4} + \frac{\mu q}{2}; \tag{7.4b}$$

$$\dot{p}|_4 = -\frac{Kp\sin\tau}{2} + \frac{Kq\cos\tau}{2} + \frac{1}{16}\gamma\mu p^3 - \frac{\mu p^3}{8} + \frac{\mu p}{2} - \frac{3}{16}\gamma\mu p q^2 - \frac{1}{8}\mu p q^2, \tag{7.5a}$$

$$\dot{q}|_4 = -\frac{Kp\cos\tau}{2} - \frac{Kq\sin\tau}{2} - \frac{3}{16}\gamma\mu p^2 q - \frac{1}{8}\mu p^2 q + \frac{1}{16}\gamma\mu q^3 - \frac{\mu q^3}{8} + \frac{\mu q}{2}. \tag{7.5b}$$

Here again subscripts 2 and 4 refer respectively to the cases corresponding to $\Omega = 2$ and $\Omega = 4$. Fig. 7.2(b) exhibits the limit cycles that are not perfect circles about the origin in p-q plane. Thus, it is clear that for either of the cases, the slow variation of the limit cycle amplitude is manifested through the slow variation of the distance of the phase point on the closed trajectory from the origin in p-q plane.

Now, we ask the question if the system allows for bi-rhythmicity. We realize that a convenient way to search for it is to look for stable fixed points (except the one at the origin) and stable limit cycles in the corresponding p-q plane. A closer look at Eqs. (7.4) and (7.5) reveals that $(0,0)$ is a common fixed point and, additionally, we have seen that they possess limit cycles. Straightforward linear stability analysis about the fixed point for the case $\Omega = 4$ yields $\left(\mu \pm iKe^{\pm i\tau}\right)/2$ as the eigenvalues that clearly has real negative part and there is no local bifurcation possible with change in γ. In fact, detailed numerical study suggests that, for the appropriately fixed parameters and $\Omega = 4$, no changes occur except that the oscillation in the amplitude of the limit cycle becomes less perceptible with increase in γ. Naturally, one expects only mono-rhythmicity in the system.

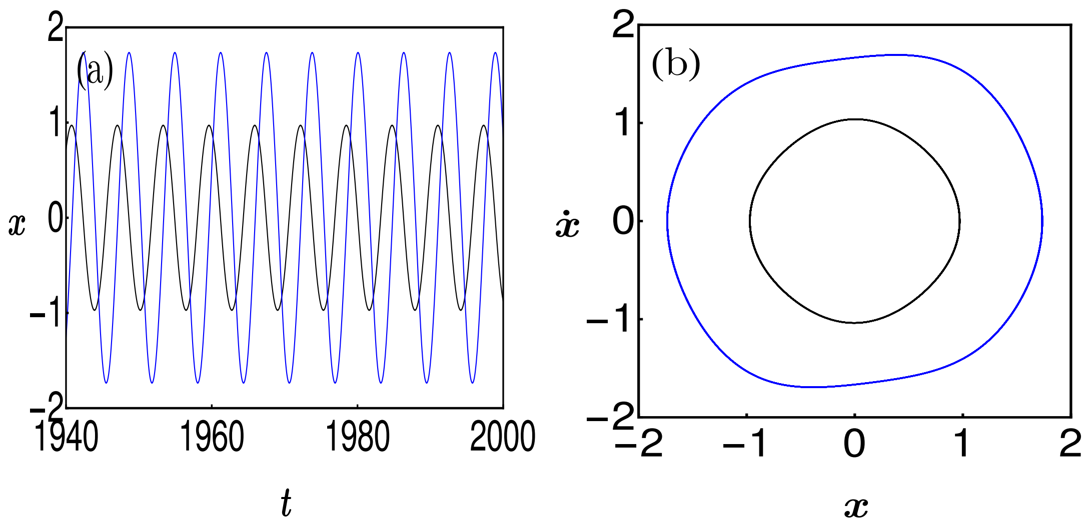

Figure 7.4: *Bi-rhythmic response of PENVO with delay.* The time series plot (a) and the phase space plot (b) for Eq. (7.1) with $\gamma = 3.3$, $K = \mu = 0.1$, $\tau = 0.623$, and $\Omega = 2$. The blue and the black lines correspond to two different initial conditions.

The case of $\Omega = 2$ is, however, very interesting: The linear stability about $(0, 0)$ yields the eigenvalues $(\pm\sqrt{\gamma^2\mu^2 - 2K^2\cos(2\tau) - 2K^2} - 2K\sin\tau + 2\mu)/4$ and thus the character of the fixed point can change with the value of γ, e.g., it is quite clear that for small values of γ (other parameters being appropriately fixed) the origin should be a focus and for larger values it should be a saddle. The full study of Eq. (7.4) being analytically quite cumbersome, we present a numerical illustration of how bi-rhythmicity is generated by varying γ.

In this respect, please see Fig. 7.3 where we have depicted the vector plots corresponding to Eq. (7.4) for $\gamma = 1.5$, $\gamma = 2.5$ and $\gamma = 3.3$. We have fixed $\Omega = 2$, $\tau = 0.623$, and $K = \mu = 0.1$. Careful study reveals that, as γ is increased, after $\gamma \approx 1.82$ the origin becomes a saddle from an unstable focus. The saddle however is born along with two stable foci (say, F_1^- and F_1^+) at which the stable manifolds of the saddle terminate; two other stable foci are also born (say, F_2^- and F_2^+) and the limit cycle, that exists around the origin for $\gamma \lesssim 1.82$, is annihilated. One observes that at a given γ, the value of $p^2 + q^2$ is same for F_1^- and F_1^+, and also for F_2^- and F_2^+, meaning that only two (and not four) different limit cycles can be observed in the PENVO with delay when $\gamma \gtrsim 1.82$. We verify this conclusion by numerically solving Eq. (7.1) for two different initial conditions but at the same set of parameter values and as shown in Fig. 7.4, we observe bi-rhythmic oscillations. To conclude what we have shown is that by changing γ we can induce bi-rhythmicity or conversely, one can say that if the system is already bi-rhythmic, *we can make the system mono-rhythmic by using γ as a control parameter.*

Up to now we have seen how a delay term added in the PENVO modifies the antiresonance and the resonance at $\Omega = 2$ and $\Omega = 4$ respectively, and furthermore, gives rise to bi-rhythmicity that in turn can be controlled by the strength of the periodically modulated nonlinearity in PENVO. Another natural modification of the Van der Pol oscillator with multiple limit cycles is a variant of the Van der Pol oscillator—originally proposed [55, 85] to model enzyme reaction in biochemical system—with a sextic order polynomial as damping coefficient:

$$\ddot{x} + \mu(-1 + x^2 - \alpha x^4 + \beta x^6)\dot{x} + x = 0. \tag{7.6}$$

Here, $0 < \mu \ll 1$ and $\alpha, \beta > 0$. We call it Kaiser oscillator. It has three concentric limit cycles surrounding an unstable focus at the origin: two of them are stable and the unstable one acts as the boundary separating the basins of attractions of the two stable cycles. However, whether there are two stable limit cycles (bi-rhythmicity) or only one (mono-rhythmicity) strictly depends on values of α and β. Under the assumption that $\mu \ll 1$, straightforward application of the K-B method helps to demarcate the regions of bi-rhythmicity and mono-rhythmicity in $\alpha - \beta$ parameter space (see Fig. D.1 in Appendix D.1). In the context of this chapter, it is of immediate curiosity to ponder upon the important questions like 'can one find resonance and antiresonance in the Kaiser oscillator', 'would periodically modulating the nonlinearity control the inherent bi-rhythmicity in the Kaiser oscillator', etc.

The addition of the periodic modulation of nonlinearity in the Kaiser oscillator get us the following equation:

$$\ddot{x} + \mu\left[1 + \gamma\cos(\Omega t)\right](-1 + x^2 - \alpha x^4 + \beta x^6)\dot{x} + x \quad = 0, \tag{7.7}$$

where $\gamma > 0$. For obvious reasons, henceforth we aptly call this system: multicycle PENVO. Again, the K-B method yields,

$$\dot{\bar{r}} \quad = \quad \frac{1}{128}\bar{r}\mu\left(-5\beta\bar{r}^6 + 8\alpha\bar{r}^4 - 16\bar{r}^2 + 64\right) + A_\Omega(\bar{r}, \bar{\phi}; \gamma), \tag{7.8a}$$

$$\dot{\bar{\phi}} \quad = \quad B_\Omega(\bar{r}, \bar{\phi}; \gamma) + O(\mu^2). \tag{7.8b}$$

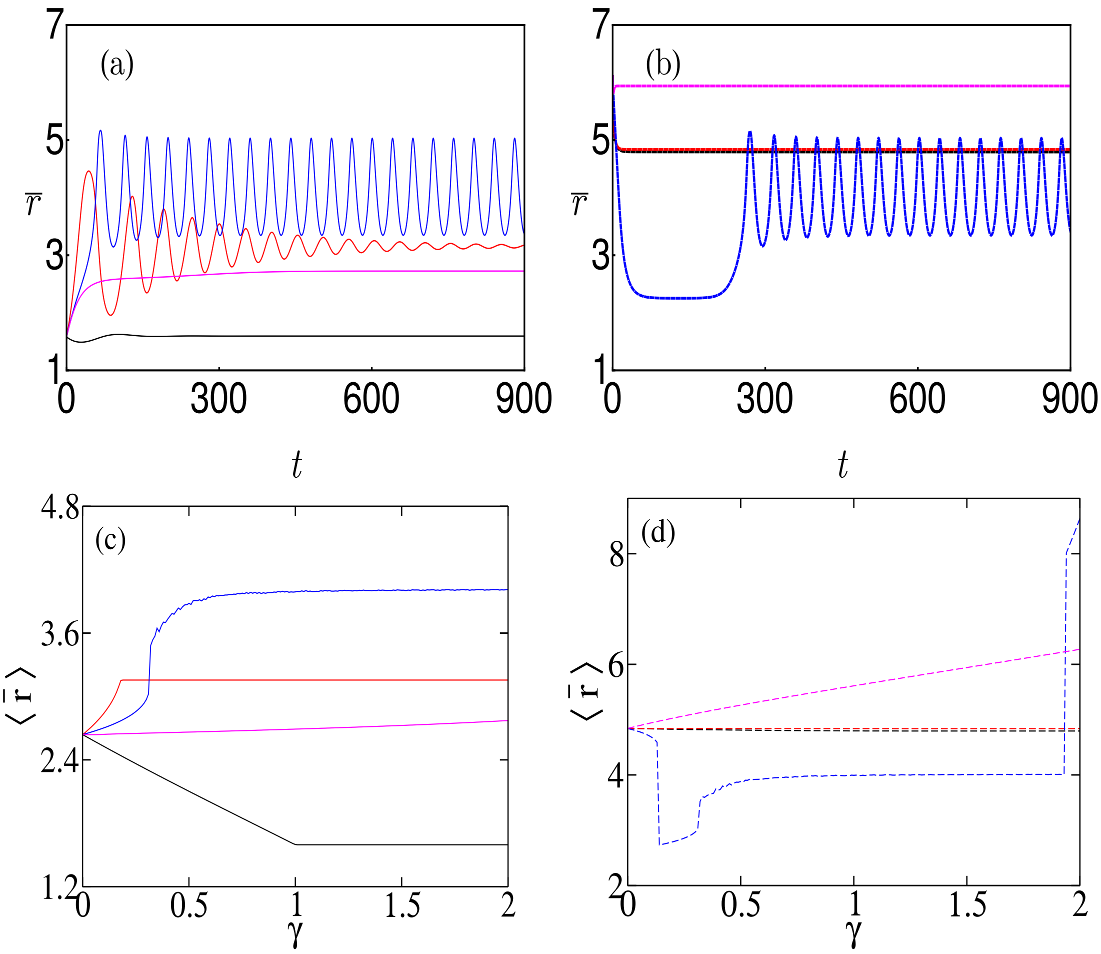

Figure 7.5: *Resonant and antiresonant responses in multicycle PENVO.* Presented are time series plots (subplot a and b) corresponding to both small (solid line) and large (dotted line) cycles for $\Omega = 2$(black), 4(red), 6(blue) and 8(magenta). Furthermore, subplots (c) and (d) depict how the averaged amplitudes of the responses change with $\gamma \in [0, 2]$. It is depicted that the smaller limit cycle shows resonances for the case $\Omega = 4$, 6 and 8 but antiresonance for the case $\Omega = 2$; the larger limit cycle admits resonance for $\Omega = 8$ but antiresonance for the case $\Omega = 2$, 4 and 6. The values of the parameters used to numerically solve Eq. (7.8) for the purpose of the figure are $\alpha = 0.144$, $\beta = 0.005$, $\mu = 0.1$ and $\gamma = 1.5$ (in subplot a and b).

Here the symbols are in their usual meaning as detailed in section 7.2. The subscripts specify the value of Ω at which A_Ω and B_Ω have to be determined; the functions have singularities at $\Omega = 2$, 4, 6 and 8, and their limiting values at these Ω-values are respectively,

$$
\begin{aligned}
A_2 &= -\frac{1}{64}\gamma\bar{r}\mu\cos(2\bar{\phi})\left(\beta\bar{r}^6 - \alpha\bar{r}^4 + 16\right), && (7.9a)\\
B_2 &= -\frac{1}{64}\gamma\mu\sin(\bar{\phi})\cos(\bar{\phi})\left(7\beta\bar{r}^6 - 10\alpha\bar{r}^4 + 16\bar{r}^2 - 32\right); && (7.9b)\\
A_4 &= \frac{1}{64}\gamma\bar{r}^3\mu\cos(4\bar{\phi})\left(\beta\bar{r}^4 - 2\alpha\bar{r}^2 + 4\right), && (7.9c)\\
B_4 &= -\frac{1}{128}\gamma\bar{r}^2\mu\sin(4\bar{\phi})\left(7\beta\bar{r}^4 - 8\alpha\bar{r}^2 + 8\right); && (7.9d)\\
A_6 &= -\frac{1}{64}\gamma\bar{r}^5\mu\cos(6\bar{\phi})\left(\alpha - \beta\bar{r}^2\right), && (7.9e)\\
B_6 &= \frac{1}{128}\gamma\bar{r}^4\mu\sin(6\bar{\phi})\left(2\alpha - 3\beta\bar{r}^2\right); && (7.9f)\\
A_8 &= \frac{1}{256}\beta\gamma\bar{r}^7\mu\cos(8\bar{\phi}), && (7.9g)\\
B_8 &= -\frac{1}{256}\beta\gamma\bar{r}^6\mu\sin(8\bar{\phi}). && (7.9h)
\end{aligned}
$$

As before, we go on to p-q plane to recast set of equations (7.8) for all four Ω-values in terms of p and q variables (see Appendix D.2) in order to understand the dynamics conveniently. For all the four values of Ω, the origin—p, q=(0,0)—is a fixed point that on doing linear stability analysis, turns out to be unstable for all values of γ. Since now the corresponding equations of motion are much more cumbersome to handle analytically, we resort to a numerical investigation of the systems. First however we need to pick appropriate value of α and β. We choose $\alpha = 0.144$ and $\beta = 0.005$ that would allow the Kaiser oscillator (multicycle PENVO with $\gamma = 0$) to exhibit bi-rhythmicity (see Appendix D.1); the amplitudes of the limit cycles that are concentric circles about $(x, \dot{x}) = (0, 0)$ in the limit $\mu \to 0$ are approximately 2.64 and 4.84 respectively. In what follows, we work with $\mu = 0.1$.

We now turn on the periodic modulation of the nonlinear term, i.e., we work with the multicycle PENVO with non-zero γ. We scan the system for various values of γ and present the results for γ up to 2 in Fig. 7.5. For illustrative purpose, consider $\gamma = 1.5$. We note that the amplitude of the smaller limit cycle of the Kaiser oscillator increases for the case $\Omega = 4$, 6 and 8 (resonances) but decreases for the case $\Omega = 2$ (antiresonance). Similarly, while the amplitude of the larger limit cycle of the Kaiser oscillator increases for the case $\Omega = 8$ (resonance), but it decreases for the case $\Omega = 2$, 4 and 6 (antiresonances). As an aside, for the case $\Omega = 6$, we also note that the amplitudes of both the cycles themselves oscillate and the

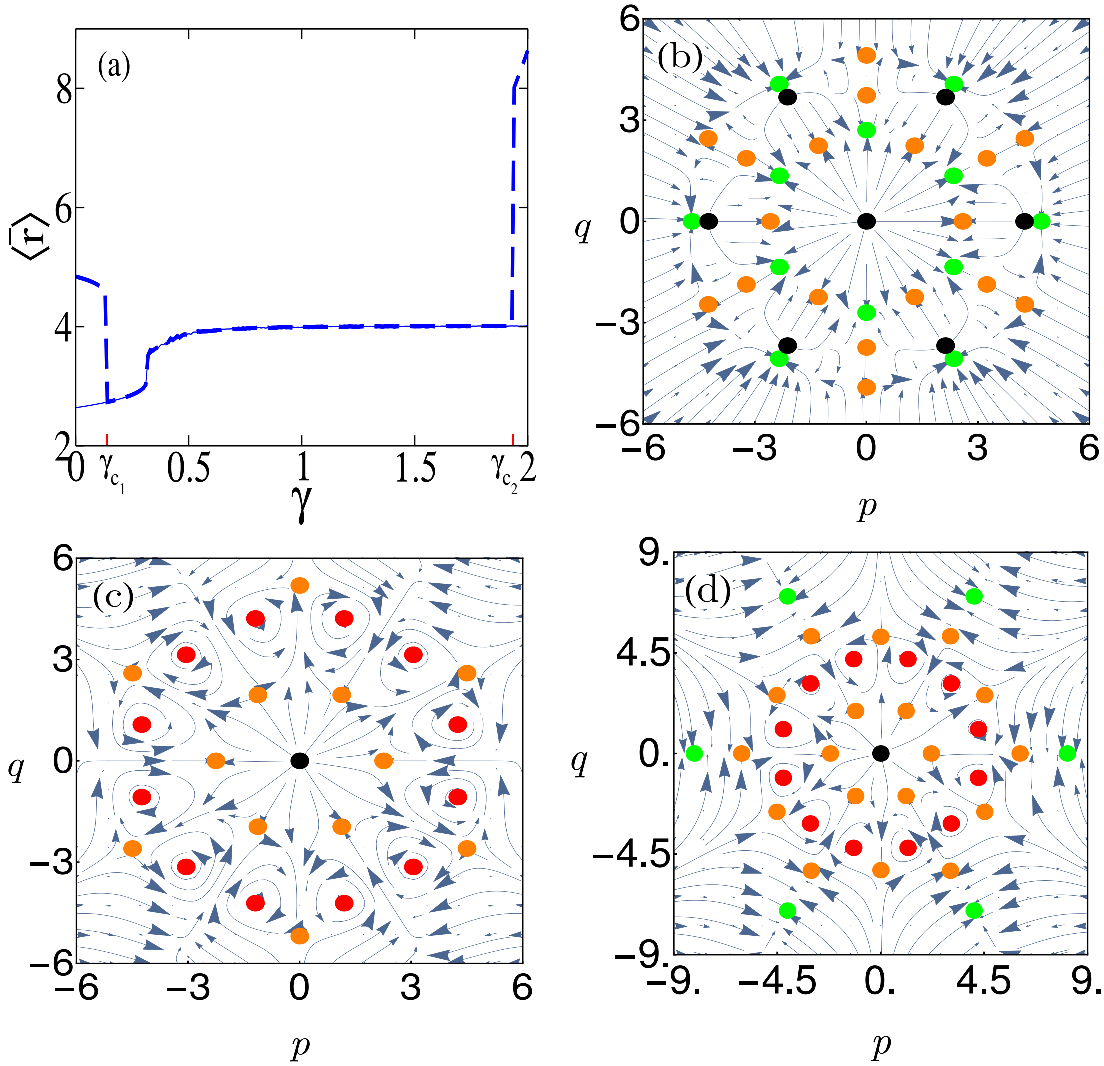

Figure 7.6: *Strength of periodic modulation of nonlinear damping controls bi-rhythmicity in multicycle PENVO.* Subplot (a) presents the observation that the average amplitudes of the periodic responses—the smaller limit cycle (solid blue line) and the larger limit cycle (dotted blue line)—merge for an intermediate range of γ between $\gamma_{c_1} \approx 0.138$ to $\gamma_{c_2} \approx 1.935$ resulting in mono-rhythmicity. Streamplots (b)-(d) depict repellers [unstable node (black dot) , unstable focus (red dot) and saddle (orange dot)] and attractors [stable node (green dot) and stable limit cycle (around each red dot; not explicitly shown)] in p-q space of the multicycle PENVO at $\gamma = 0.1,\ 1.5,$ and 1.95, respectively. Other parameter values have been fixed at $\alpha = 0.144,\ \beta = 0.005,\ \mu = 0.1$ and $\Omega = 6$. In subplot (b), there are two sets of stable foci with two distinct values of $\sqrt{p^2+q^2}$ (hence bi-rhythmicity), while in subplot (c) only attractor (and hence mono-rhythmicity) is a limit cycle—a circle that passes through all the unstable foci with same $\sqrt{p^2+q^2}$-values and centred at origin. In subplot (c), in addition to this limit cycle, another set of stable foci appear with same $\sqrt{p^2+q^2}$-value (hence bi-rhythmicity).

response corresponding to the outer limit cycle changes from antiresonance to resonance as γ increases (see Fig. 7.5d).

More interesting, however, is the fact that the resonance and the antiresonance, manifested as limit cycles with oscillating amplitudes, for $\Omega = 6$ merge—as implicitly shown in Fig. 7.6(a)—for a range of γ-values: $\gamma \in (\gamma_{c_1}, \gamma_{c_2}) \approx (0.138, 1.935)$. This means that *$\gamma$ is yet again acting as a control parameter in bringing about mono-rhythmicity by suppressing the bi-rhythmicity.* To understand the phase dynamics of control of the aforementioned bi-rhythmicity, we consider the system (7.8) in $(p,\ q)$ plane at three representative values of γ, viz., $\gamma = 0.1$ (Fig. 7.6b), $\gamma = 1.5$ (Fig. 7.6c), and $\gamma = 1.95$ (Fig. 7.6d). For $\gamma = 0.1 < \gamma_{c_1}$, a case of bi-rhythmicity, there are twelve stable nodes—the only attractors in the phase space—that can be classified into two groups such that one group of nodes has $\sqrt{p^2+q^2} \approx 2.70$ and the other group has $\sqrt{p^2+q^2} \approx 4.67$. This corresponds to the fact that there are two distinct limit-cycles in the x-$\dot{x}$ plane, and their radii are 2.70 and 4.67; in other words, the system is bi-rhythmic. In the mono-rhythmic case of $\gamma = 1.5 \in (\gamma_{c_1}, \gamma_{c_2})$, we note that the attractors now are twelve limit cycles whose centers (unstable focus) lie on a circle of radius 4.38 (approximately). Thus, the system has now become mono-rhythmic and the limit cycle in the x-$\dot{x}$ plane has periodically oscillating amplitude. The bifurcation leading to the creation of the twelve symmetrically placed limit cycles takes place at $\gamma = \gamma_{c_1}$ when the stable nodes and the unstable saddles (present at $\gamma < \gamma_{c_1}$) merge appropriately to give rise to the limit cycles (seen at $\gamma > \gamma_{c_1}$). Finally, For $\gamma = 1.95 > \gamma_{c_2}$, the system showcases bi-rhythmic behaviour yet again: the six symmetrically placed asymptotically stable nodes in the corresponding p-q plane have identical values for $\sqrt{p^2+q^2}$, *viz.*, 8.12 that corresponds to the amplitude of the limit cycle of the multicycle PENVO.

We note that the bi-rhythmicity present at other resonance and antiresonance conditions, i.e., for $\Omega = 2$, 4, and 8, could not be controlled to mono-rhythmicity by the variation in γ. However, recalling that in section 7.2 the combination of γ and delay could effect control of bi-rhythmicity, one is tempted to add delay term, viz., '$-K(t-\tau)$' in the left hand side of Eq. (7.7) with a hope to effect control of bi-rhythmicity for $\Omega = 2$, 4, and 8. The introduction on such a delay term in the Kaiser oscillator shifts the region of bi-rhythmicity in the α-β plane (see Appendix D.1). In the simultaneous presence of non-zero γ and K, the multicycle PENVO's response at $\Omega = 2$, 4, 6, and 8 can be analyzed using the K-B method just as has been done in detail for Eq. (7.1) and Eq. (7.7). We omit the repetitive details and rather present the summary of the analyzes in Fig. 7.7(a-b). We note that the delay does indeed suppress bi-rhythmicity; and interestingly in the case of $\Omega = 8$, *γ can be seen to be a control parameter even in the presence of delay.*

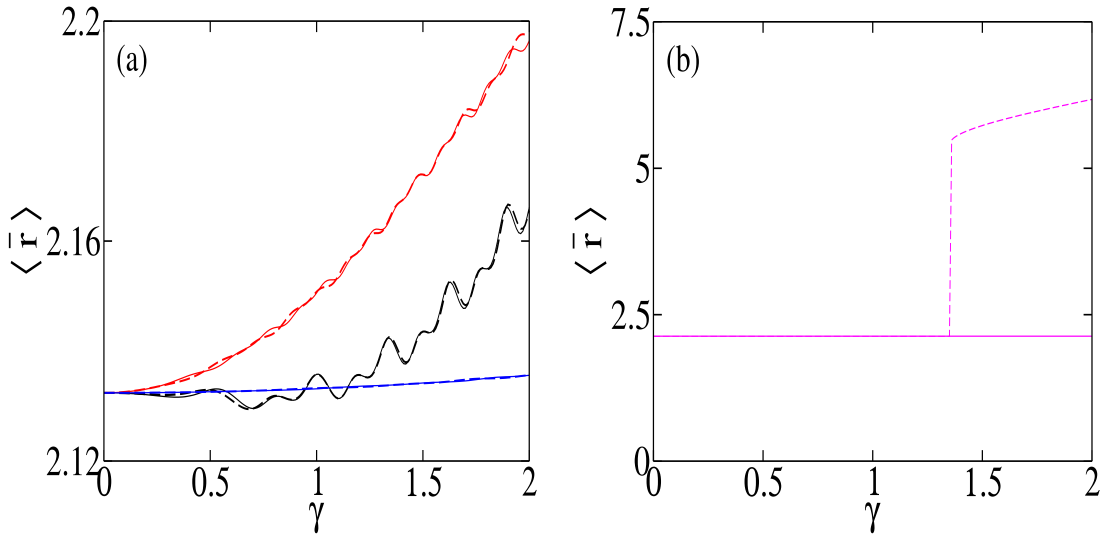

Figure 7.7: *Controlling bi-rhythmicity via delay in multicycle PENVO.* Subplots (a) and (b) exhibit how the averaged amplitudes change with $\gamma \in [0, 2]$ corresponding to both small (solid line) and large (dotted line) cycles for $\Omega = 2$ (black), 4 (red), 6 (blue) and 8 (magenta). The values of the relevant parameters used in the figure are $\alpha = 0.144$, $\beta = 0.005$, $\mu = 0.1$ and $\tau = 0.2$. We note that the responses are mostly mono-rhythmic.

7.4 CONCLUSION

How to control bi-rhythmicity in an oscillator is an interesting question. In this chapter we have illustrated that the bi-rhythmicity seen in the delayed Van der Pol oscillator and the Van der Pol oscillator modified to have higher order nonlinear damping (the Kaiser oscillator) can be suppressed if the nonlinear terms of the oscillators are periodically modulated. This periodic modulation of the nonlinear damping also brings about resonance and antiresonance responses in the aforementioned oscillators. In order to characterize the responses, we have presented perturbative calculations using the K-B method and supplemented them with ample numerical solutions for the systems of ordinary differential equations under consideration. We have also discussed in detail how to understand the bifurcations leading to mono-rhythmicity from bi-rhythmicity (and vice versa) from the relevant phase space trajectories obtained via the perturbative technique.

We recall that the introduction of delay is one of the popularly known method of controlling bi-rhythmicity. However, as we have seen in section 7.2, delay can introduce bi-rhythmicity as well. It is interesting to realize in such cases periodically modifying the nonlinear terms can change the bi-rhythmic behaviour to mono-rhythmic. A comparison of responses due to delay and parametric excitation in a limit cycle system provides an extra tool-kit for controlling bi-rhythmicity when one alone may not be fruitful. We may point out that the delay term we have used in this chapter is completely position dependent as opposed to the more commonly investigated velocity dependent delay terms [56, 91, 92] in the literature.

We strongly believe that the proposed idea of controlling multi-rhythmicity by invoking periodic modulation of nonlinear terms could be useful in plethora of limit cycle systems. However, we do not believe that building a general universal mechanism behind this phenomenon can be proposed easily; each system has to be analyzed on a case-by-case basis.

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A

$\lambda - \omega$ FORM OF VAN DER POL—DUFFING OSCILLATOR

A.1 VAN DER POL—DUFFING OSCILLATOR IN $\lambda - \omega$ FORM

A two-dimensional kinetic set of equations can be cast into a Liénard–Levinson–Smith (LLS) form if there exists any linear transformation and one can apply the reverse of it to get back the original set of equations. A question may arise, is, for a given LLS system— $\ddot{x} + f(x, \dot{x})\dot{x} + g(x)$, what will be its other kinetic form called $\lambda - \omega$ version in addition to the trivial one: $\dot{x} = y,\ \dot{y} = -f(x, \dot{x})\dot{x} - g(x)$?

To answer this question we explore a general two-dimensional oscillator model with a $\lambda - \omega$ version having the following kinetic form:

$$\begin{aligned} \frac{dx}{dt} &= \lambda(r)x - \omega(r)y, \\ \frac{dy}{dt} &= \omega(r)x + \lambda(r)y; \quad r = \sqrt{x^2 + y^2}. \end{aligned} \tag{A.1}$$

The above system is well known for the existence of Hopf bifurcation and limit cycle attractor with amplitude r. If any transformation is introduced to change the coordinates from cartesian to polar $((x, y) \to (r, \phi))$ s.t. $(x, y) = (r\cos\phi,\ r\sin\phi)$ with $(r, \phi) = \left(\sqrt{x^2 + y^2}, tan^{-1}\frac{y}{x}\right)$ then one can easily get

$$\begin{aligned} \frac{dr}{dt} &= r\lambda(r), \\ \frac{d\phi}{dt} &= \omega(r). \end{aligned} \tag{A.2}$$

The non-zero root(s), say, $r = R$, of the amplitude equation confirms the existence of limit cycle(s). The phase solution becomes then $\phi = \phi_0 + \omega(R)t$ of frequency $\omega(R)$.

Now, to answer the question through an example, we consider the Van der Pol–Duffing system,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x - \epsilon x^3 = 0. \tag{A.3}$$

With the amplitude and phase equation using Krylov–Bogolyubov averaging we have,

$$\begin{aligned} \dot{\overline{r}} &= \frac{\epsilon}{2}\left(\overline{r} - \frac{\overline{r}^3}{4}\right), \\ \dot{\overline{\phi}} &= -\frac{3\,\epsilon}{8}\overline{r}^3. \end{aligned} \tag{A.4}$$

One can segregate the above into the form as,

$$\begin{aligned} \dot{\overline{r}} &= \overline{r}\frac{\epsilon}{8}\left(4 - \overline{r}^2\right) = \overline{r}\lambda(\overline{r}); \quad & \lambda(\overline{r}) &= \frac{\epsilon}{8}\left(4 - \overline{r}^2\right), \\ \dot{\overline{\phi}} &= \omega(\overline{r}); & \omega(\overline{r}) &= -\frac{3\,\epsilon}{8}\overline{r}^3. \end{aligned} \tag{A.5}$$

Now, if we consider a two-dimensional kinetic $\lambda - \omega$ system then it looks like,

$$\begin{aligned} \frac{dx}{dt} &= \frac{\epsilon}{8}\left(4 - \overline{r}^2\right)x - \left(-\frac{3\,\epsilon}{8}\overline{r}^3\right)y, \\ \frac{dy}{dt} &= \left(-\frac{3\,\epsilon}{8}\overline{r}^3\right)x + \frac{\epsilon}{8}\left(4 - \overline{r}^2\right)y; \quad \overline{r} = \sqrt{x^2 + y^2}. \end{aligned} \tag{A.6}$$

Simulation of Eq. (A.6) agrees with Eq. (A.3) has a limit cycle of radius ≈ 2(like Van der Pol). There is a structural difference between them; the cycle of system (A.6) is circular whereas the structure of the cycle of system (A.3) is not exactly circular. As a conclusion one can say that weak nonlinearity of the system can provide better agreement as orbits of such weakly LLS systems (Eq. A.3) shows more circular in nature (closer to harmonic oscillator solution).

Conversely, if we have a $\lambda - \omega$ type kinetic form then we can construct a LLS form of it even if there does not exist any linear transformation in the route of casting into LLS system. To explain this we can have a hypothesis: for a LLS system, if there exists a limit cycle attractor (repeller) then we have shown that the amplitude equation contains the parts which are in the damping force function (say, $f(x, \dot{x})$) and the phase equation contains the parts that are in the restoring force function (say, $g(x)$) iff $f(x, \dot{x}) = f_1(x^2, \dot{x}^2)$. So, keeping this in mind, if we have the $\lambda - \omega$ form (A.1) having the amplitude and phase equations (A.2) then without loss of generality we can assume the LLS form of it as

$$\ddot{x} + \lambda(r)\dot{x} + x + \omega(r) = 0. \tag{A.7}$$

B

LOTKA-VOLTERRA SYSTEM: LIÉNARD–LEVINSON–SMITH (LLS) FORM

To obtain the LLS form of Lotka-Volterra System[10, 28, 36],

$$\begin{aligned} \dot{x}(t) &= \alpha x - \beta x y, \\ \dot{y}(t) &= -\gamma y + \delta x y, \end{aligned} \tag{B.1}$$

let us set $z = \delta x + \beta y$ then $\dot{z} = \alpha\delta x - \beta\gamma y = u \implies x = \frac{\dot{z}+\gamma z}{(\alpha+\gamma)\delta}$ and $y = \frac{-\dot{z}+\alpha z}{(\alpha+\gamma)\beta}$. After taking t derivative upon $\dot{z}$ one can have,

$$\ddot{z} \quad = (\alpha - \gamma)\dot{z} + \alpha\gamma z + \tfrac{\dot{z}^2}{\alpha+\gamma} + \tfrac{\gamma-\alpha}{\alpha+\gamma} z\dot{z} - \tfrac{\alpha\gamma}{\alpha+\gamma} z^2. \tag{B.2}$$

The fixed point $(0, 0)$ gives a saddle solution which is not of any interest in the present context. Choosing the remaining non-zero fixed point for further investigations and after taking perturbation $z = \xi + z_s$ around the fixed point $z_s = \alpha + \gamma = \delta x_s + \beta y_s \neq 0$, one can get the LLS form with $F(\xi, \dot{\xi}) = a_1\xi + a_2\dot{\xi}$ with $a_1 = \frac{\alpha-\gamma}{\alpha+\gamma}$ and $a_2 = -\frac{1}{\alpha+\gamma}$. It is to be noted that $G(\xi)$ contains nonlinearity with $G(\xi) = \omega^2\xi + a_3\xi^2$ where $\omega = \sqrt{\alpha\gamma} = Im(\lambda)(+ve$ sense) and $a_3 = \frac{\alpha\gamma}{\alpha+\gamma}$. After introducing a small parameter ϵ_1 in the constants, a_i, b_i such that $a_i = \epsilon_1 b_i, i = 1, 2, 3$ the above equation reduces to

$$\ddot{\xi} + \epsilon_1(b_1\xi + b_2\dot{\xi})\dot{\xi} + \omega^2\xi + \epsilon_1 b_3\xi^2 = 0, \tag{B.3}$$

and after rescaling time, t by τ with $\tau = \omega t$ (s.t. $\xi(t)$ changes to $Z(\tau)$), the above equation takes the form, $\ddot{Z}(\tau) + \epsilon h(Z(\tau), \dot{Z}(\tau)) + Z(\tau) = 0$ with $h = (k_1 Z + k_2\dot{Z})\dot{Z} + k_3 Z^2$ where $k_1 = \omega b_1$, $k_2 = \omega^2 b_2$, $k_3 = b_3$, and $\epsilon = \frac{\epsilon_1}{\omega^2}$. We have considered $0 < \epsilon \ll 1$, which means $0 < \epsilon_1 \ll \omega^2 = \alpha\gamma \leq 1 \implies \alpha \leq \frac{1}{\gamma}$, since $\alpha, \gamma > 0$.

C

GENERATION OF BI-RHYTHMICITY IN OTHER SYSTEMS AND $\lambda - \omega$ VERSION OF IT

C.1 BI-RHYTHMICITY IN SCHNAKENBERG MODEL

The governing kinetic equations for the original Schnakenberg model[40] are,

$$\begin{aligned} \dot{x} &= a + x^2 y - x, \\ \dot{y} &= b - x^2 y. \end{aligned} \tag{C.1}$$

Our aim is to remodel the above mono-rhythmic system into a bi-rhythmic one by casting into a Liénard–Levinson–Smith (LLS) form by adding extra nonlinearity. To transform the above system of equations into a LLS form, defining a suitable linear transformation as $z = x + y - (p + q)$, s.t. $\dot{z} = u$ holds, where $u = a + b - x$. The inverse transformations provides $x = a + b - u$ and $y = z + u - (a + b) + (p + q)$. The point $(p, q) = \left(a + b, \frac{b}{(a+b)^2}\right)$ is the fixed point of (C.1). Derivative w.r.t. time again provides the generalised Rayleigh (or LLS form), as,

$$\begin{aligned} \ddot{z} + \{f_{00} - 2(a + b)z + (p + q - 3(a + b) + z)\dot{z} + \dot{z}^2\}\dot{z} + \omega^2 z = 0; \\ \omega = (a + b),\ f_{00} = a^2 + 2a\left(\frac{1}{a + b} + b\right) + b^2 - 1 \end{aligned} \tag{C.2}$$

In some recent developments[40, 44, 87] in this direction of Liénard approach, we have shown that for a locally stable limit cycle, the condition $f_{00} < 0$ must be satisfied. This condition helps to find the parameter region for mono-rhythmicity and for a fixed value of (a, b) it is able to provide the mono-rhythmicity in the LLS system (C.2) as well as original kinetic system (C.1). The parameter space for the mono-rhythmicity is given in Fig. C.1(a) with the shaded area for mono-rhythmicity and for a specific set of values of $(a, b) = (0.001422, 0.9806)$, the corresponding phase space for the LLS system (C.2) is given in Fig. C.1(b). The original phase space for Schnakenberg mono-rhythmic model (C.1) is given in Fig. C.2, where continuous and dotted curves are the plots for system (C.1) and its projection (fixed point shifting) to the origin, respectively. The fixed point shifting is taken to get an idea about the radius of the limit cycle that can be estimated by the axes cuts[259]. For better estimation one needs to go through perturbation theory with second order correction[259]. So, if we shift the system (C.1) from a non-zero fixed point $(x_s, y_s) = (p, q)$ to the

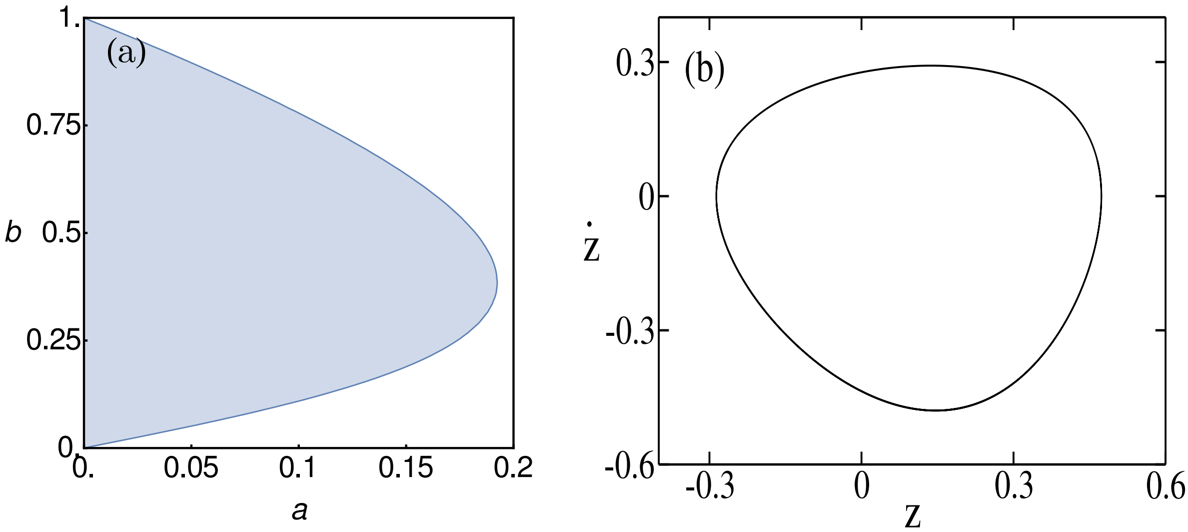

Figure C.1: *Mono-rhythmic Schnakenberg model:* The plot (a) is the parameter regime of Eq. (C.1) for a stable limit cycle and (b) phase space of the LLS form (C.2) of the Schnakenberg model (C.1) representing a stable limit cycle for the parameter values $(a, b) = (0.001422, 0.9806)$.

the origin by the transformations $x = x + p$ and $y = y + q$ then the corresponding kinetic flow equations take the form,

$$\begin{aligned} \dot{x} &= \frac{x\left(b(b+x)-a^2\right)}{(a+b)^2} + y(a+b+x)^2, \\ \dot{y} &= -y(a+b+x)^2 - \frac{bx(2(a+b)+x)}{(a+b)^2}. \end{aligned} \tag{C.3}$$

with the cycle (see Fig. C.2) cuts the x-axis at ≈ 0.36 where as the y-axis at ≈ 0.48.

As we have seen in the chapter 6, that the bi-rhythmicity can be constructed with the addition of higher order terms in the polynomial to the LLS system. To fulfil the aim towards the direction of the construction of bi-rhythmicity for the Schnakenberg model we have to add higher order nonlinear polynomial function of z, say, $-\alpha z^4 + \beta z^6$, (or $-\alpha \dot{z}^4 + \beta \dot{z}^6$, discussed later in section C.1.1) to the damping force function which helps us to have higher order amplitude equation and hence bi-rhythmicity. So, the generalised Rayleigh oscillator (C.2) can be rewritten as,

$$\ddot{z} + \{f_{00} - 2(a+b)z + (p+q-3(a+b)+z)\dot{z} + \dot{z}^2 - \alpha z^4 + \beta z^6\}\dot{z} + \omega^2 z = 0. \tag{C.4}$$

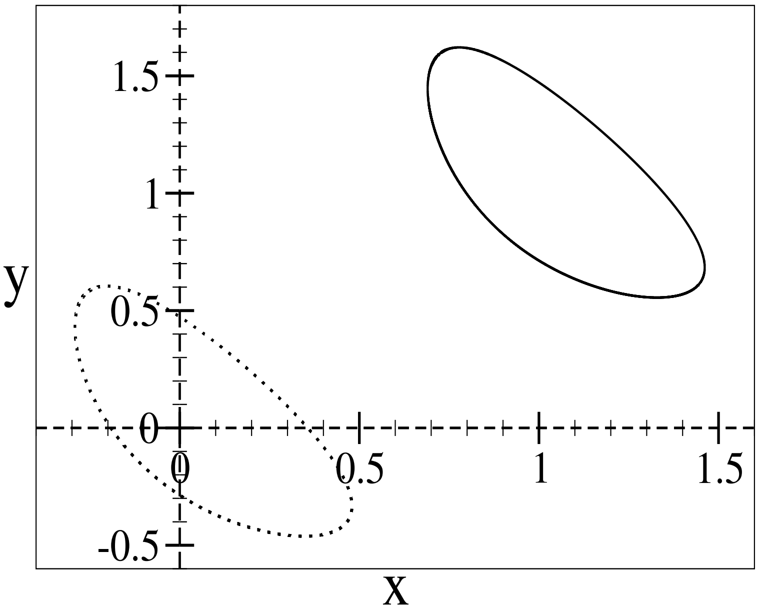

Figure C.2: *Mono-rhythmic Schnakenberg model:* Phase space plots of mono-rhythmic Schnakenberg model in $(x,\ y)$ coordinate system where the thin curve is for the original kinetic equations (C.1) with a non-zero fixed point and the bold one is the projection of the system to the origin can be explained by the set of kinetic equations (C.3). Here, the parameter values are $a = 0.001422$ and $b = 0.9806$.

To implement perturbative analysis in the above LLS system (C.4) to have amplitude equation as well as the bi-rhythmic parameter zone, let us rescale the original time (say, t) by $\omega t \to \tau$. Then the dependent variable of system (C.4) is transformed into $\xi(\tau)$ from $z(t)$ and system (C.4) can be written as,

$$\ddot{\xi} + \epsilon \left[\frac{1}{\sigma} \{ f_{00} - 2(a+b)\xi + (p+q-3(a+b)+\xi)\omega\dot{\xi} + \omega^2\dot{\xi}^2 - \alpha\xi^4 + \beta\xi^6 \} \right] \dot{\xi}$$
$$+\xi = 0, \tag{C.5}$$

where, $\epsilon = \frac{\sigma}{\omega}$ and $\sigma = |f_{00}|$, and restricted the system by $0 < \epsilon \ll 1$ to apply perturbative analysis through the system parameters (a, b). Now the system is ready to perform any perturbative analysis.

Here, we apply Renormalisation Group (RG) method to have amplitude and phase equation. So, by introducing a perturbative solution of ξ as, $\xi = \xi_0 + \epsilon\ \xi_1 + O(\epsilon^2)$, system (C.5) yields,

$$\begin{aligned} \epsilon^0 : \ddot{\xi}_0 + \xi_0 &= 0, && \text{(C.6)} \\ \epsilon^1 : \ddot{\xi}_1 + \xi_1 &= -\frac{1}{\sigma}\{\dot{\xi}_0 \left(-2(a+b)\xi_0 + f_{00} - \alpha{\xi_0}^4 + \beta{\xi_0}^6 \right) \\ &\quad + \omega{\dot{\xi}_0}^2 \left(-3a - 3b + p + q + \xi_0 \right) + \omega^2{\dot{\xi}_0}^3 \}. && \text{(C.7)} \end{aligned}$$

Setting initial conditions, $\xi(\tau = 0) = H$ and $\dot{\xi}(\tau = 0) = 0$ provides $\xi_0(0) = H$ and $\xi_i(0) = 0, \forall i > 0$ along with $\dot{\xi}_i(0) = 0, \forall i \geq 0$. By applying these initial conditions one finds a harmonic oscillator solution $\xi_0(\tau) = H\cos(\tau + \theta_0)$ having amplitude H and phase θ_0 for the zeroth order correction. The zeroth order correction admits the solution of ξ_1, the first order solution (the complete method is given in the overview). Finally, one finds the approximate analytical solution of $\xi = \xi_0 + \epsilon\xi_1$ (upto first order correction), is,

$$\begin{aligned}
\xi &= H\cos(\tau+\theta_0)\\
&+\epsilon\Bigg[\frac{5\beta\sin(\tau-\theta_0)H^7}{256\sigma}+\frac{5\beta\sin(\tau+\theta_0)H^7}{256\sigma}+\frac{9\beta\sin(\tau+3\theta_0)H^7}{256\sigma}+\frac{5\beta\sin(\tau+5\theta_0)H^7}{512\sigma}+\frac{\beta\sin(\tau+7\theta_0)H^7}{768\sigma}+\frac{9\beta\sin(\tau-3\theta_0)H^7}{512\sigma}\\
&+\frac{5\beta\sin(\tau-5\theta_0)H^7}{768\sigma}+\frac{\beta\sin(\tau-7\theta_0)H^7}{1024\sigma}-\frac{5\beta\tau\cos(\tau+\theta_0)H^7}{128\sigma}-\frac{9\beta\sin(3(\tau+\theta_0))H^7}{512\sigma}-\frac{5\beta\sin(5(\tau+\theta_0))H^7}{1536\sigma}\\
&-\frac{\beta\sin(7(\tau+\theta_0))H^7}{3072\sigma}+\frac{\alpha\tau\cos(\tau+\theta_0)H^5}{16\sigma}+\frac{3\alpha\sin(3(\tau+\theta_0))H^5}{128\sigma}+\frac{\alpha\sin(5(\tau+\theta_0))H^5}{384\sigma}-\frac{\alpha\sin(\tau-\theta_0)H^5}{32\sigma}-\frac{\alpha\sin(\tau+\theta_0)H^5}{32\sigma}\\
&-\frac{3\alpha\sin(\tau+3\theta_0)H^5}{64\sigma}-\frac{\alpha\sin(\tau+5\theta_0)H^5}{128\sigma}-\frac{3\alpha\sin(\tau-3\theta_0)H^5}{128\sigma}-\frac{\alpha\sin(\tau-5\theta_0)H^5}{192\sigma}+\frac{\omega\cos(\tau-\theta_0)H^3}{16\sigma}+\frac{\omega\cos(\tau+3\theta_0)H^3}{16\sigma}\\
&+\frac{3\omega^2\sin(\tau-\theta_0)H^3}{16\sigma}+\frac{3\omega^2\sin(\tau+\theta_0)H^3}{16\sigma}+\frac{\omega^2\sin(3(\tau+\theta_0))H^3}{32\sigma}-\frac{3\tau\omega^2\cos(\tau+\theta_0)H^3}{8\sigma}-\frac{\tau\omega\sin(\tau+\theta_0)H^3}{8\sigma}\\
&-\frac{\omega\cos(\tau+\theta_0)H^3}{16\sigma}-\frac{\omega^2\sin(\tau+3\theta_0)H^3}{16\sigma}-\frac{\omega\cos(3(\tau+\theta_0))H^3}{32\sigma}-\frac{\omega\cos(\tau-3\theta_0)H^3}{32\sigma}-\frac{\omega^2\sin(\tau-3\theta_0)H^3}{32\sigma}+\frac{3a\omega H^2}{2\sigma}\\
&+\frac{3b\omega H^2}{2\sigma}+\frac{p\omega\cos(\tau)H^2}{2\sigma}+\frac{q\omega\cos(\tau)H^2}{2\sigma}+\frac{a\omega\cos(2(\tau+\theta_0))H^2}{2\sigma}+\frac{b\omega\cos(2(\tau+\theta_0))H^2}{2\sigma}+\frac{p\omega\cos(\tau+2\theta_0)H^2}{4\sigma}\\
&+\frac{q\omega\cos(\tau+2\theta_0)H^2}{4\sigma}+\frac{a\omega\cos(\tau-2\theta_0)H^2}{4\sigma}+\frac{b\omega\cos(\tau-2\theta_0)H^2}{4\sigma}+\frac{a\sin(2(\tau+\theta_0))H^2}{3\sigma}+\frac{b\sin(2(\tau+\theta_0))H^2}{3\sigma}\\
&-\frac{p\omega H^2}{2\sigma}-\frac{q\omega H^2}{2\sigma}-\frac{3a\omega\cos(\tau)H^2}{2\sigma}-\frac{3b\omega\cos(\tau)H^2}{2\sigma}-\frac{a\sin(\tau+2\theta_0)H^2}{2\sigma}-\frac{b\sin(\tau+2\theta_0)H^2}{2\sigma}-\frac{3a\omega\cos(\tau+2\theta_0)H^2}{4\sigma}\\
&-\frac{3b\omega\cos(\tau+2\theta_0)H^2}{4\sigma}-\frac{p\omega\cos(2(\tau+\theta_0))H^2}{6\sigma}-\frac{q\omega\cos(2(\tau+\theta_0))H^2}{6\sigma}-\frac{a\sin(\tau-2\theta_0)H^2}{6\sigma}-\frac{b\sin(\tau-2\theta_0)H^2}{6\sigma}\\
&-\frac{p\omega\cos(\tau-2\theta_0)H^2}{12\sigma}-\frac{q\omega\cos(\tau-2\theta_0)H^2}{12\sigma}+\frac{f_{00}\sin(\tau-\theta_0)H}{4\sigma}+\frac{f_{00}\sin(\tau+\theta_0)H}{4\sigma}-\frac{f_{00}\tau\cos(\tau+\theta_0)H}{2\sigma}\Bigg]
\end{aligned}$$

As we are applying RG method which is predominantly a multi-scale perturbative method where the direct time interval $(\tau - 0)$ can have to split into $(\tau - s) + (s - 0)$ to observe the slow time variation $(0 < s \ll \tau)$ of amplitudes and phases. Further, to absorb the divergence in the amplitude and phase solution, they have to modify from (H, θ_0) to $(H(s), \theta(s))$ with the transformations $H(s) = \frac{H}{W_1(s,0)}$ and $\theta(s) = \theta_0 - W_2(s, 0)$, where $W_1(s, 0) = 1 + \sum_1^\infty \epsilon^n p_n$ and $W_2(s, 0) = 0 + \sum_1^\infty \epsilon^n q_n$. The terms including $O(\epsilon^2)$ can be neglected due to first order analysis and one finds $(W_1,\ W_2) = (1 + \epsilon p_1,\ \epsilon q_1)$. So, after setting the above considerations in the solution ξ and removing the secular terms we get,

$$
\begin{aligned}
\xi &= H(s)\cos(\tau+\theta(s)) \\
&+\epsilon\Bigg[\frac{5\beta\sin(\tau-\theta(s))H(s)^7}{256\sigma}+\frac{5\beta\sin(\tau+\theta(s))H(s)^7}{256\sigma}+\frac{9\beta\sin(\tau+3\theta(s))H(s)^7}{256\sigma}+\frac{5\beta\sin(\tau+5\theta(s))H(s)^7}{512\sigma} \\
&+\frac{\beta\sin(\tau+7\theta(s))H(s)^7}{768\sigma}+\frac{9\beta\sin(\tau-3\theta(s))H(s)^7}{512\sigma}+\frac{5\beta\sin(\tau-5\theta(s))H(s)^7}{768\sigma}+\frac{\beta\sin(\tau-7\theta(s))H(s)^7}{1024\sigma} \\
&-\frac{5\beta\tau\cos(\tau+\theta(s))H(s)^7}{128\sigma}-\frac{9\beta\sin(3(\tau+\theta(s)))H(s)^7}{512\sigma}-\frac{5\beta\sin(5(\tau+\theta(s)))H(s)^7}{1536\sigma}-\frac{\beta\sin(7(\tau+\theta(s)))H(s)^7}{3072\sigma} \\
&+\frac{\alpha\tau\cos(\tau+\theta(s))H(s)^5}{16\sigma}+\frac{3\alpha\sin(3(\tau+\theta(s)))H(s)^5}{128\sigma}+\frac{\alpha\sin(5(\tau+\theta(s)))H(s)^5}{384\sigma}-\frac{\alpha\sin(\tau-\theta(s))H(s)^5}{32\sigma}-\frac{\alpha\sin(\tau+\theta(s))H(s)^5}{32\sigma} \\
&-\frac{3\alpha\sin(\tau+3\theta(s))H(s)^5}{64\sigma}-\frac{\alpha\sin(\tau+5\theta(s))H(s)^5}{128\sigma}-\frac{3\alpha\sin(\tau-3\theta(s))H(s)^5}{128\sigma}-\frac{\alpha\sin(\tau-5\theta(s))H(s)^5}{192\sigma}+\frac{\omega\cos(\tau-\theta(s))H(s)^3}{16\sigma} \\
&+\frac{\omega\cos(\tau+3\theta(s))H(s)^3}{16\sigma}+\frac{3\omega^2\sin(\tau-\theta(s))H(s)^3}{16\sigma}+\frac{3\omega^2\sin(\tau+\theta(s))H(s)^3}{16\sigma}+\frac{\omega^2\sin(3(\tau+\theta(s)))H(s)^3}{32\sigma} \\
&-\frac{3\tau\omega^2\cos(\tau+\theta(s))H(s)^3}{8\sigma}-\frac{\tau\omega\sin(\tau+\theta(s))H(s)^3}{8\sigma}-\frac{\omega\cos(\tau+\theta(s))H(s)^3}{16\sigma}-\frac{\omega^2\sin(\tau+3\theta(s))H(s)^3}{16\sigma} \\
&-\frac{\omega\cos(3(\tau+\theta(s)))H(s)^3}{32\sigma}-\frac{\omega\cos(\tau-3\theta(s))H(s)^3}{32\sigma}-\frac{\omega^2\sin(\tau-3\theta(s))H(s)^3}{32\sigma}+\frac{3a\omega H(s)^2}{2\sigma}+\frac{3b\omega H(s)^2}{2\sigma}+\frac{p\omega\cos(\tau)H(s)^2}{2\sigma} \\
&+\frac{q\omega\cos(\tau)H(s)^2}{2\sigma}+\frac{a\omega\cos(2(\tau+\theta(s)))H(s)^2}{2\sigma}+\frac{b\omega\cos(2(\tau+\theta(s)))H(s)^2}{2\sigma}+\frac{p\omega\cos(\tau+2\theta(s))H(s)^2}{4\sigma} \\
&+\frac{q\omega\cos(\tau+2\theta(s))H(s)^2}{4\sigma}+\frac{a\omega\cos(\tau-2\theta(s))H(s)^2}{4\sigma}+\frac{b\omega\cos(\tau-2\theta(s))H(s)^2}{4\sigma}+\frac{a\sin(2(\tau+\theta(s)))H(s)^2}{3\sigma} \\
&+\frac{b\sin(2(\tau+\theta(s)))H(s)^2}{3\sigma}-\frac{p\omega H(s)^2}{2\sigma}-\frac{q\omega H(s)^2}{2\sigma}-\frac{3a\omega\cos(\tau)H(s)^2}{2\sigma}-\frac{3b\omega\cos(\tau)H(s)^2}{2\sigma}-\frac{a\sin(\tau+2\theta(s))H(s)^2}{2\sigma} \\
&-\frac{b\sin(\tau+2\theta(s))H(s)^2}{2\sigma}-\frac{3a\omega\cos(\tau+2\theta(s))H(s)^2}{4\sigma}-\frac{3b\omega\cos(\tau+2\theta(s))H(s)^2}{4\sigma}-\frac{p\omega\cos(2(\tau+\theta(s)))H(s)^2}{6\sigma} \\
&-\frac{q\omega\cos(2(\tau+\theta(s)))H(s)^2}{6\sigma}-\frac{a\sin(\tau-2\theta(s))H(s)^2}{6\sigma}-\frac{b\sin(\tau-2\theta(s))H(s)^2}{6\sigma}-\frac{p\omega\cos(\tau-2\theta(s))H(s)^2}{12\sigma} \\
&-\frac{q\omega\cos(\tau-2\theta(s))H(s)^2}{12\sigma}+\frac{f_{00}\sin(\tau-\theta(s))H(s)}{4\sigma}+\frac{f_{00}\sin(\tau+\theta(s))H(s)}{4\sigma}-\frac{f_{00}\tau\cos(\tau+\theta(s))H(s)}{2\sigma}\Bigg]
\end{aligned}
\tag{C.8}
$$

According to the RG scheme, the final solution based on the combination of slow-fast time variable cannot be dependent upon the arbitrary time scale s i.e. $\left(\frac{\partial\xi}{\partial s}\right)_\tau = 0$ which leads to

$$
\begin{aligned}
\frac{dH(s)}{ds} &= -\frac{\epsilon}{128\sigma}H\left(64f_{00}+48H^2\omega^2-8\alpha H^4+5\beta H^6\right)+O(\epsilon^2) \quad \text{and} \\
\frac{d\theta(s)}{ds} &= \frac{\epsilon\omega}{8\sigma}H^2+O(\epsilon^2),
\end{aligned}
\tag{C.9}
$$

where the notations are their usual meanings. The above amplitude equation can show the mono-rhythmic behaviour for $\alpha = \beta = 0$ for a fixed value of (a, b), say, $(0.001422, 0.9806)$ (from the parameter region Fig. C.1a) along with an approximate amplitude of LLS system (C.2) as well as the original Schnakenberg model (C.1). The respective plots are shown in Fig. C.1(b) and Fig. C.2, respectively, where the amplitude is ≈ 0.212748 which is more accurately reflected in the Liénard zone than original kinetic phase plot.

Now, to get bi-rhythmicity, the bi-rhythmic parameter region of (α, β) must have to find out by keeping unchanged parameter values of (a, b) (which are taken from mono-rhythmic parameter zone). In this consideration the amplitude equation provides the parameter space for (α, β) which is shown in Fig. C.3(a). Then the values of (α, β), say, $(9.89762, 6.93655)$ from the obtained region gives the amplitudes of the bi-rhythmic oscillation where the amplitudes for the oscillation are, $\approx$ 0.222115, 0.982666 and 1.12607 which are of stable, unstable and stable, respectively. But the bi-rhythmicity is not reflecting in the phase plane if we solve Eq. (C.4).

Here, one problem may arise owing to the higher order nonlinearity in the the system i.e. Eq. (C.4). In the chapter 6, we have already shown that, to have more and more rhythmicity in a LLS system that can be captured in the phase space, the system must contain higher order polynomials along with a nonlinearity-tuning parameter which has to be weaker than a mono-rhythmic oscillator. In the analyzed system we don't have any artificial nonlinearity controlling parameter. So, to capture the claimed bi-rhythmic situation in the phase plane, introducing a nonlinearity-control parameter (say, μ) through the multiplicative adjustment to damping force function. So, to capture the real scenario, reconstructing the LLS system from (C.5), as,

$$\ddot{z} + \mu\{f_{00} - 2(a+b)z + (p+q-3(a+b)+z)\dot{z} + \dot{z}^2 - \alpha z^4 + \beta z^6\}\dot{z} + \omega^2 z = 0, \tag{C.10}$$

and after rescaling we have,

$$\ddot{\xi} + \mu\epsilon\left[\frac{1}{\sigma}\{f_{00} - 2(a+b)\xi + (p+q-3(a+b)+\xi)\omega\dot{\xi} + \omega^2\dot{\xi}^2 - \alpha\xi^4 + \beta\xi^6\}\right]\dot{\xi} + \xi = 0. \tag{C.11}$$

Doing similar perturbative analysis we have the following amplitude and phase equations:

$$\begin{aligned} \frac{dH(s)}{ds} &= -\frac{\mu\epsilon}{128\sigma}H\left(64f_{00} + 48H^2\omega^2 - 8\alpha H^4 + 5\beta H^6\right) + O(\mu^2), \\ \frac{d\theta(s)}{ds} &= \frac{\mu\epsilon\omega}{8\sigma}H^2 + O(\mu^2). \end{aligned} \tag{C.12}$$

The above equations are almost similar to Eq. (C.9) other than an extra multiplicative nonlinearity-control parameter μ and the magnitudes of the amplitudes do not get affected by it, and we get almost the same result as in above. The corresponding phase space plot of Eq. (C.10) is given in Fig.C.3(b) for a particular set of parameters, say, $\alpha = 9.89762$ and $\beta = 6.93655$. The parameter μ is tuned at 0.1 and for the smaller value of it provides orbits more circular in the phase plane. So, bi-rhythmicity of Eq. (C.4) does not reflect in the phase space until an extra parameter is added in the system.

As we already have the linear transformations which are the path to the construction of LLS system (C.4) from the kinetic flow equations (C.1), so the reverse transformations must help us to get back to the set of autonomous kinetic equations (C.15) from the LLS

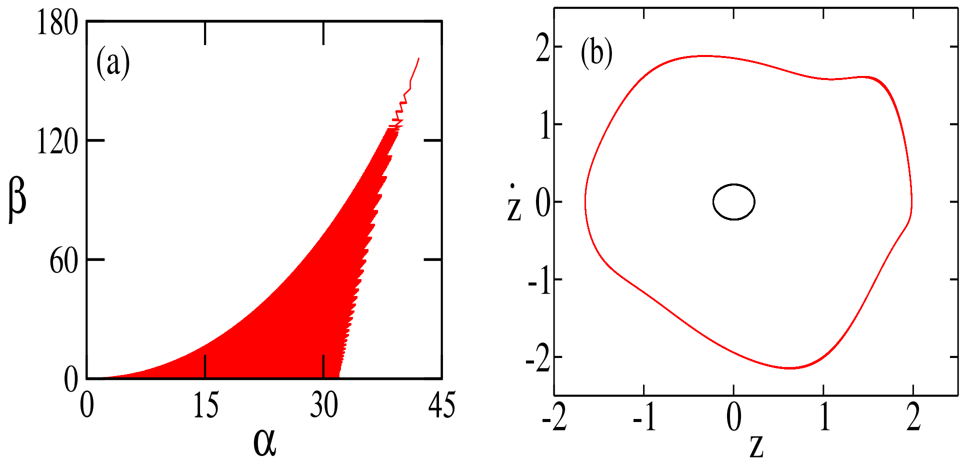

Figure C.3: *Bi-rhythmic Schnakenberg model:* (a) Parameter region for $(\alpha,\ \beta)$ to have bi-rhythmicity for the LLS form of the extended Schnakenberg model (C.4) and (b) shows the corresponding bi-rhythmic phase space. Here, the parameter values are $a = 0.001422,\ b = 0.9806,\ \alpha = 9.89762,\ \beta = 6.93655$ and $\mu = 0.1$.

system (C.10). The system characteristics remain invariant in both the situations as linear transformations are applied here. Then the reverse transformations provides:

$$\begin{aligned} x &= a + b - u \\ \Longrightarrow \dot{x} &= -\dot{u} = -\dot{u}|_{(x,y)} = -\dot{z}|_{\{z=x+y-(p+q),\ u=a+b-x\}} \qquad (C.13) \\ y &= z + u - (a+b) + (p+q) \\ \Longrightarrow \dot{y} &= \dot{z} + \dot{u} = a + b - x + \dot{u}|_{(x,y)} = a + b - x + \dot{z}|_{\{z=x+y-(p+q),\ u=a+b-x\}} \qquad (C.14) \end{aligned}$$

Applying above steps for Eq. (C.10), we can have the following simplified two-dimensional autonomous coupled form of kinetic equations:

$$\begin{aligned} \dot{x} = \mu(a+b-x)\Bigg(& a^2 + 2(a+b)(p+q-x-y) + (a+b-x)(-3(a+b)+x+y) + (a+b-x)^2 \\ &+ 2a\left(\frac{1}{a+b}+b\right) + b^2 - \alpha(p+q-x-y)^4 + \beta(p+q-x-y)^6 - 1\Bigg) + (a+b)^2(-p-q+x+y), \end{aligned}$$

$$\begin{aligned} \dot{y} = -\mu(a+b-x)\Bigg(& a^2+2(a+b)(p+q-x-y)+(a+b-x)(-3(a+b)+x+y)+(a+b-x)^2+2a\left(\frac{1}{a+b}+b\right) \\ &+ b^2 - \alpha(p+q-x-y)^4 + \beta(p+q-x-y)^6 - 1\Bigg) + (a+b)^2(p+q-x-y) + a + b - x. \end{aligned} \qquad (C.15)$$

The fixed point (i.e. $(x_s, y_s) = (p, q)$) of (C.1) remain unchanged for the bi-rhythmic Schnakenberg model (C.15) and if we shift the fixed point $(x_s, y_s) = (p, q)$ to the origin using the transformation $x = x + p$ and $y = y + q$ then system (C.15) is transformed into the following set of equations:

$$\dot{x} = a^2(-\mu x + x + y) + 2a\left(\mu x\left(y - \frac{1}{a+b}\right) + b(-\mu x + x + y)\right) + b\mu x\left(\frac{x}{(a+b)^2} + 2y\right)$$
$$+ b^2(-\mu x + x + y) + \mu x\left(-\beta x^6 - 6\beta x^5 y + x^4\left(\alpha - 15\beta y^2\right) + 4x^3 y\left(\alpha - 5\beta y^2\right)\right.$$
$$\left. + x^2\left(6\alpha y^2 - 15\beta y^4\right) + x\left(-6\beta y^5 + 4\alpha y^3 + y\right) - \beta y^6 + \alpha y^4 + 1\right),$$

$$\dot{y} = x\left(a^2(\mu-1) + a\left(2\mu\left(\frac{1}{a+b} - y\right) + 2b(\mu-1)\right) + b^2(\mu-1) - 2b\mu y - \mu + \beta\mu y^6 - \alpha\mu y^4 - 1\right)$$
$$+ \mu x^2\left(-\frac{b}{(a+b)^2} + 6\beta y^5 - 4\alpha y^3 - y\right) - y(a+b)^2 + \beta\mu x^7 + 6\beta\mu x^6 y$$
$$- \mu x^5\left(\alpha - 15\beta y^2\right) + 4\mu x^4 y\left(5\beta y^2 - \alpha\right) + 3\mu x^3 y^2\left(5\beta y^2 - 2\alpha\right). \tag{C.16}$$

The corresponding bi-rhythmic phase portrait is given in Fig. C.4.

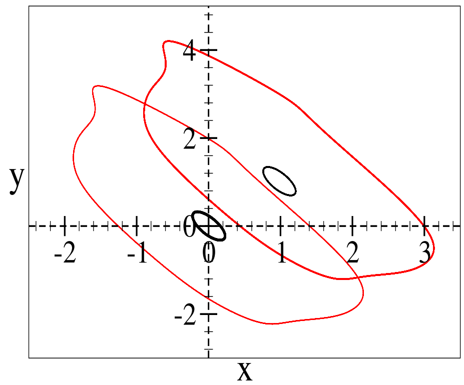

Figure C.4: *Bi-rhythmic Schnakenberg model:* Phase space plots in the $(x,\ y)$ coordinate system where the thin curves are the plots for Eq. (C.15) (having non-zero fixed point) and the bold curves are for Eq. (C.16) (containing trivial fixed point) is the projection to the origin of Eq. (C.15). Here, the parameter values are $a = 0.001422$, $b = 0.9806$, $\alpha = 9.89762$, $\beta = 6.93655$ and $\mu = 0.1$.

C.1.1 *Alternative situation*

Now, if we extend the Schnakenberg bi-rhythmic model by a velocity dependent nonlinearity $-\alpha\dot{z}^4 + \beta\dot{z}^6$ instead of position dependent nonlinearity $-\alpha z^4 + \beta z^6$ by keeping unchanged the parameter values of a and b (are, 0.001422 and 0.9806) then equation (C.10) looks like,

$$\ddot{z} + \mu\{f_{00} - 2(a+b)z + (p + q - 3(a+b) + z)\dot{z} + \dot{z}^2 - \alpha\dot{z}^4 + \beta\dot{z}^6\}\dot{z} + \omega^2 z = 0, \tag{C.17}$$

and then after rescaling we have,

$$\ddot{\xi} + \mu\epsilon\left[\frac{1}{\sigma}\{f_{00} - 2(a+b)\xi + (p+q-3(a+b)+\xi)\omega\dot{\xi} + \omega^2\dot{\xi}^2 - \alpha\omega^4\dot{\xi}^4 + \beta\omega^6\dot{\xi}^6\}\right]\dot{\xi} + \xi = 0. \quad \text{(C.18)}$$

So, after applying RG analysis one obtains,

$$\begin{aligned} \frac{dH(s)}{ds} &= -\frac{\mu\epsilon H\left(64f_{00} + 35\beta H^6\omega^6 - 40\alpha H^4\omega^4 + 48H^2\omega^2\right)}{128\sigma} + O(\mu^2) \quad \text{and} \\ \frac{d\theta(s)}{ds} &= \frac{\mu\epsilon\omega}{8\sigma}H^2 + O(\mu^2), \end{aligned} \quad \text{(C.19)}$$

From the above expressions of (C.19), the amplitude equation provides the bi-rhythmic parameter region of $\alpha - \beta$ which is shown in Fig. C.5(a) and the respective phase space plots are given in Fig. C.5(b).

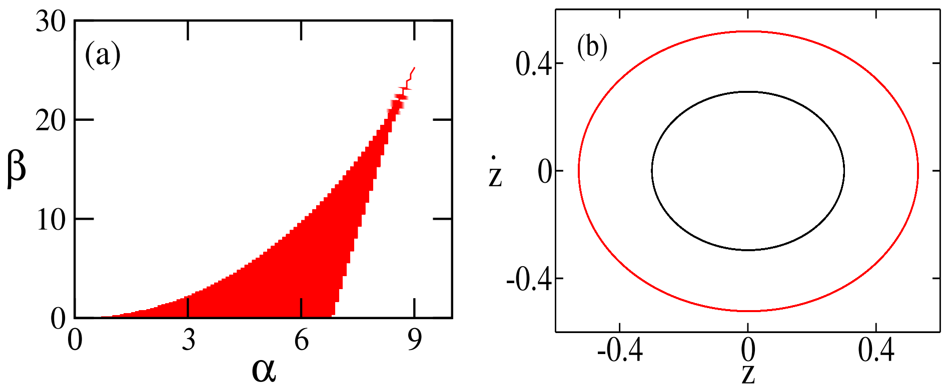

Figure C.5: *Bi-rhythmic Schnakenberg model (Rayleigh type extension):* Plot (a) shows the parameter region for Schnakenberg bi-rhythmic oscillator and (b) shows the corresponding bi-rhythmic phase space plots for a fixed set of parameter values $a = 0.001422$, $b = 0.9806$, $\alpha = 8.24548$, $\beta = 19.2277$ and $\mu = 0.01$.

Finally, inverse transformations help us to have the kinetic flow equations in the (x, y) plane as

$$\begin{aligned} \dot{x} = \mu(a+b-x)\Big(& a^2 + 2(a+b)(p+q-x-y) - \alpha(a+b-x)^4 + \beta(a+b-x)^6 + (a+b-x)(-3(a+b)+x+y) \\ & + (a+b-x)^2 + 2a\left(\frac{1}{a+b}+b\right) + b^2 - 1\Big) + (a+b)^2(-p-q+x+y), \end{aligned}$$

$$\begin{aligned} \dot{y} = -\mu(a+b-x)\Big(& a^2 + 2(a+b)(p+q-x-y) - \alpha(a+b-x)^4 + \beta(a+b-x)^6 + (a+b-x)(-3(a+b)+x+y) \\ & + (a+b-x)^2 + 2a\left(\frac{1}{a+b}+b\right) + b^2 - 1\Big) + (a+b)^2(p+q-x-y) + a + b - x. \end{aligned} \quad \text{(C.20)}$$

Shifting the fixed point of the above system to the origin, it reduces to,

$$
\begin{aligned}
\dot{x} &= a^2(-\mu x + x + y) + 2a\left(\mu x\left(y - \frac{1}{a+b}\right) + b(-\mu x + x + y)\right) \\
&\quad + b\mu x\left(\frac{x}{(a+b)^2} + 2y\right) + b^2(-\mu x + x + y) + \mu x\left(-\beta x^6 + \alpha x^4 + xy + 1\right),
\end{aligned}
$$

$$
\begin{aligned}
\dot{y} &= x\left(a^2(\mu - 1) + a\left(2\mu\left(\frac{1}{a+b} - y\right) + 2b(\mu - 1)\right) + b^2(\mu - 1) - 2b\mu y - \mu - 1\right) \\
&\quad + x^2\left(\mu(-y) - \frac{b\mu}{(a+b)^2}\right) - y(a+b)^2 + \beta\mu x^7 - \alpha\mu x^5.
\end{aligned} \tag{C.21}
$$

C.2 BI-RHYTHMIC $\lambda - \omega$ SYSTEM

Previously we have seen that amplitude equation is able to say about multi-rhythmicity along with their multi-rhythmic oscillatiory location for a weakly nonlinear system. A mono-rhythmic $\lambda - \omega$ system (A.1) can easily be extended into a bi-rhythmic one in addition to some higher order polynomial in a systematic way and the coefficients of the higher degree polynomial reflects in the amplitude equation or vice versa. For a better demonstration, considering a general $\lambda - \omega$ system (A.1), say, $\lambda(r) = (1 - r^2)$ and $\omega(r) = 1$, then the corresponding system becomes,

$$
\begin{aligned}
\frac{dx}{dt} &= (1 - r^2)x - y, \\
\frac{dy}{dt} &= x + (1 - r^2)y,
\end{aligned} \tag{C.22}
$$

having the amplitude and phase equation,

$$
\begin{aligned}
\frac{dr}{dt} &= r(1 - r^2), \\
\frac{d\phi}{dt} &= 1.
\end{aligned} \tag{C.23}
$$

It clearly shows that the considered $\lambda - \omega$ system (C.22) has a unique stable limit cycle of amplitude 1 and the respective phase solution is, $\phi = \phi_{Initial} + t$, and the mono-rhythmicity can easily be verified by the time series as well as phase space plots which are given in Fig. C.6.

So, we have a single stable limit cycle for the above consideration of $\lambda(r)$ as it has an unique non-zero root of r. Now, if we extend the degree of the polynomial of r in the amplitude equation to have multiple distinct non-zero roots (here three) by following Blows

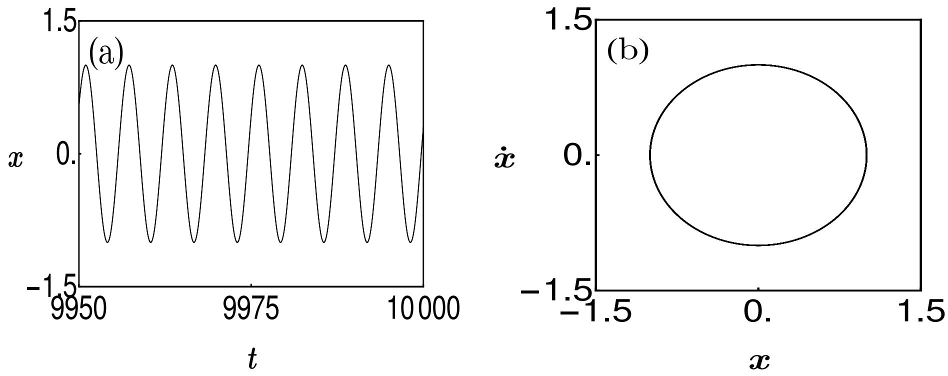

Figure C.6: *Mono-rhythmic $\lambda - \omega$ system (C.22):* Plot (a) is the time series for mono-rhythmicity and (b) shows the corresponding limit cycle.

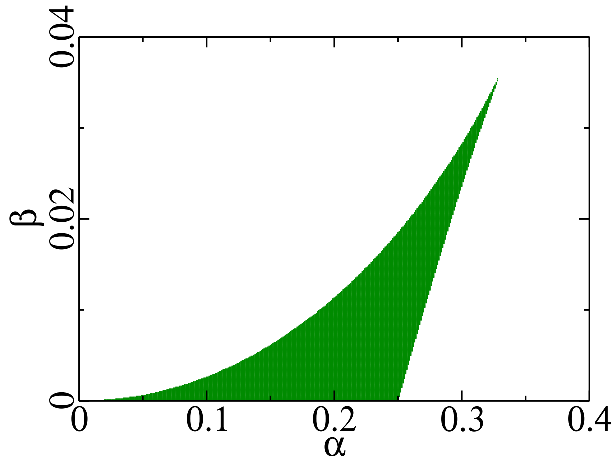

Figure C.7: *Parameter space for bi-rhythmic $\lambda - \omega$.*

theorem[272] (like Kaiser model) as $\lambda(r) = (1 - r^2 + \alpha r^4 - \beta r^6)$ then we have the following amplitude and phase equations:

$$\begin{aligned} \frac{dr}{dt} &= r(1 - r^2 + \alpha r^4 - \beta r^6), \\ \frac{d\phi}{dt} &= 1. \end{aligned} \tag{C.24}$$

After that, searching for three distinct non-zero real roots, the amplitude equation provides a parameter region for $\alpha - \beta$ is given in Fig. C.7.

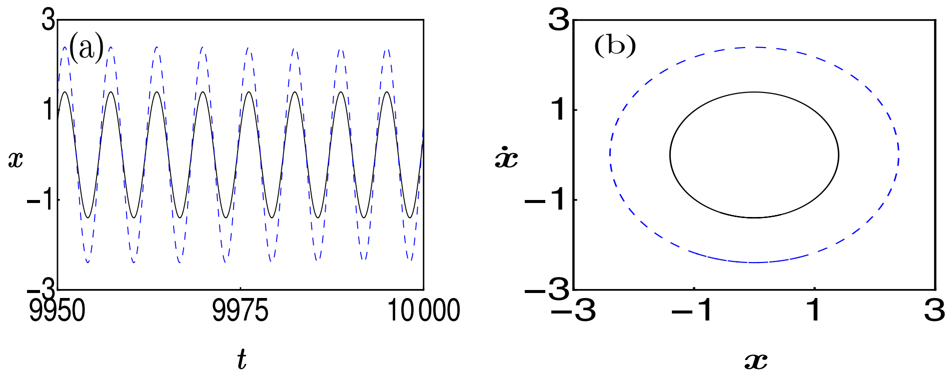

Figure C.8: *Bi-rhythmic $\lambda - \omega$ system (C.25):* Plot (a) is the time series for bi-rhythmicity and (b) shows the corresponding limit cycles.

If we choose a point from the parameter region (Fig. C.7) with $(\alpha, \beta) = (0.304681, 0.0280317)$ then we must have three distinct real roots of magnitudes, $r = 1.39833, 1.78444, 2.39365$—showing a bi-rhythmic behaviour. Finally, system of equations can be written as,

$$\begin{aligned} \frac{dx}{dt} &= (1 - (x^2 + y^2) + \alpha(x^2 + y^2)^2 - \beta(x^2 + y^2)^3)x - y, \\ \frac{dy}{dt} &= x + (1 - (x^2 + y^2) + \alpha(x^2 + y^2)^2 - \beta(x^2 + y^2)^3)y. \end{aligned} \tag{C.25}$$

This explained method can physically be examined for the creation of higher order nonlinear processes and its applicability can be verified. The respective bi-rhythmic time series and phase plots are shown in Fig. C.8 where the black curve (continuous) is for the small stable limit cycle and blue one (dashed) is for larger stable cycle.

D

BI-RHYTHMICITY IN THE KAISER OSCILLATOR: EFFECT OF DELAY

D.1 BI-RHYTHMICITY IN THE KAISER OSCILLATOR: EFFECT OF DELAY

Consider the Kaiser model in presence of a position dependent delay:

$$\ddot{x} + \mu \left(-1 + x^2 - \alpha x^4 + \beta x^6\right) \dot{x} + x - Kx(t-\tau) = 0, \tag{D.1}$$

$(0 < \epsilon, \tau \ll 1)$. When $K = 0$, the system is either mono-rhythmic or bi-rhythmic depending on the values of α and β as depicted in Fig. (D.1). It is expected that for small values of K and τ, the behaviour of the Kaiser oscillator should be qualitatively similar, although the region in the α-β plane where the bi-rhythmic behaviour is seen would be shifted slightly. This is shown in Fig. (D.1) that has been obtained by employing the Krylov–Bogolyubov method to write the equations for the amplitude as well as the phase of the system's response as

$$\dot{\bar{r}} = -\frac{\bar{r}\left(64K\sin\tau + \mu\left(5\beta\bar{r}^6 - 8\alpha\bar{r}^4 + 16\bar{r}^2 - 64\right)\right)}{128}, \tag{D.2a}$$

$$\dot{\bar{\phi}} = -\frac{1}{2}K\cos\tau, \tag{D.2b}$$

respectively. Here higher order terms have been neglected. It is clear from the existence of non-overlapping regions of bi-rhythmicity that the delay factor may transform mono-rhythmicity to bi-rhythmicity or vice versa.

D.2 FLOW EQUATIONS: MULTICYCLE PENVO WITH DELAY

On imposing parametric excitation to the nonlinearity in Eq. (D.1), we can write,

$$\ddot{x} + \mu\left[1 + \gamma\cos(\Omega t)\right]\left(-1 + x^2 - \alpha x^4 + \beta x^6\right)\dot{x} + x - Kx(t-\tau) = 0. \tag{D.3}$$

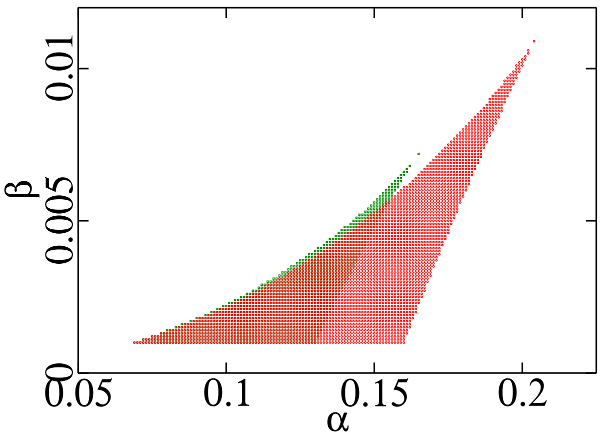

Figure D.1: *Delay changes rhythmicity.* This figure showcases for what values of α and β, systems (7.6) and (D.1) are bi-rhythmic—the green and the red zones, respectively. In other words, the changes in the bi-rhythmic zone in α-β parameter space in the presence of the time delay ($K = 0.1$ and $\tau = 0.2$) have been exhibited. Here, $\mu = 0.1$.

The corresponding amplitude and phase equations are

$$\begin{aligned} \dot{\bar{r}} &= -\frac{1}{128}\bar{r}\left(64K\sin\tau + \mu\left(5\beta\bar{r}^6 - 8\alpha\bar{r}^4 + 16\bar{r}^2 - 64\right)\right) \\ &\quad + A_\Omega(\bar{r},\bar{\phi};\gamma) + O(\mu^2); \qquad \text{(D.4a)} \\ \dot{\bar{\phi}} &= -\frac{1}{2}K\cos\tau + B_\Omega(\bar{r},\bar{\phi};\gamma) + O(\mu^2), \qquad \text{(D.4b)} \end{aligned}$$

where higher order terms have been neglected, and A_Ω and B_Ω are functions with singularities at $\Omega = 2$, 4, 6 and 8. One may resort to the L'Hôspitals' rule and go to p-q plane to rewrite the amplitude and the phase equations in terms of the coordinate of the plane:

$$
\begin{aligned}
\dot{p}_2 &= -\frac{Kp\sin\tau}{2} + \frac{Kq\cos\tau}{2} - \frac{1}{64}\beta\gamma\mu p^7 - \frac{1}{128}5\beta\mu p^7 + \frac{1}{64}\alpha\gamma\mu p^5 + \frac{1}{16}\alpha\mu p^5 + \frac{3}{32}\beta\gamma\mu p^5 q^2 - \frac{15}{128}\beta\mu p^5 q^2 - \frac{\mu p^3}{8} \\
&\quad + \frac{15}{64}\beta\gamma\mu p^3 q^4 - \frac{15}{128}\beta\mu p^3 q^4 - \frac{5}{32}\alpha\gamma\mu p^3 q^2 + \frac{1}{8}\alpha\mu p^3 q^2 - \frac{\gamma\mu p}{4} + \frac{\mu p}{2} + \frac{1}{8}\beta\gamma\mu p q^6 - \frac{5}{128}\beta\mu p q^6 - \frac{11}{64}\alpha\gamma\mu p q^4 \\
&\quad + \frac{1}{16}\alpha\mu p q^4 + \frac{1}{4}\gamma\mu p q^2 - \frac{1}{8}\mu p q^2, \\
\dot{q}_2 &= -\frac{Kp\cos\tau}{2} - \frac{Kq\sin\tau}{2} - \frac{1}{8}\beta\gamma\mu p^6 q - \frac{5}{128}\beta\mu p^6 q - \frac{15}{64}\beta\gamma\mu p^4 q^3 - \frac{15}{128}\beta\mu p^4 q^3 + \frac{11}{64}\alpha\gamma\mu p^4 q + \frac{1}{16}\alpha\mu p^4 q \\
&\quad - \frac{3}{32}\beta\gamma\mu p^2 q^5 - \frac{15}{128}\beta\mu p^2 q^5 + \frac{5}{32}\alpha\gamma\mu p^2 q^3 + \frac{1}{8}\alpha\mu p^2 q^3 - \frac{1}{4}\gamma\mu p^2 q - \frac{1}{8}\mu p^2 q + \frac{1}{64}\beta\gamma\mu q^7 - \frac{1}{128}5\beta\mu q^7 \\
&\quad - \frac{1}{64}\alpha\gamma\mu q^5 + \frac{1}{16}\alpha\mu q^5 - \frac{\mu q^3}{8} + \frac{\gamma\mu q}{4} + \frac{\mu q}{2}; \\
\dot{p}_4 &= -\frac{Kp\sin\tau}{2} + \frac{Kq\cos\tau}{2} + \frac{1}{64}\beta\gamma\mu p^7 - \frac{1}{128}5\beta\mu p^7 - \frac{1}{32}\alpha\gamma\mu p^5 + \frac{1}{16}\alpha\mu p^5 + \frac{9}{64}\beta\gamma\mu p^5 q^2 - \frac{15}{128}\beta\mu p^5 q^2 + \frac{1}{16}\gamma\mu p^3 \\
&\quad - \frac{\mu p^3}{8} - \frac{5}{64}\beta\gamma\mu p^3 q^4 - \frac{15}{128}\beta\mu p^3 q^4 - \frac{1}{16}\alpha\gamma\mu p^3 q^2 + \frac{1}{8}\alpha\mu p^3 q^2 + \frac{\mu p}{2} - \frac{13}{64}\beta\gamma\mu p q^6 - \frac{5}{128}\beta\mu p q^6 + \frac{7}{32}\alpha\gamma\mu p q^4 \\
&\quad + \frac{1}{16}\alpha\mu p q^4 - \frac{3}{16}\gamma\mu p q^2 - \frac{1}{8}\mu p q^2, \\
\dot{q}_4 &= -\frac{Kp\cos\tau}{2} - \frac{Kq\sin\tau}{2} - \frac{13}{64}\beta\gamma\mu p^6 q - \frac{5}{128}\beta\mu p^6 q - \frac{5}{64}\beta\gamma\mu p^4 q^3 - \frac{15}{128}\beta\mu p^4 q^3 + \frac{7}{32}\alpha\gamma\mu p^4 q + \frac{1}{16}\alpha\mu p^4 q \\
&\quad + \frac{9}{64}\beta\gamma\mu p^2 q^5 \frac{15}{128}\beta\mu p^2 q^5 - \frac{1}{16}\alpha\gamma\mu p^2 q^3 + \frac{1}{8}\alpha\mu p^2 q^3 - \frac{3}{16}\gamma\mu p^2 q - \frac{1}{8}\mu p^2 q + \frac{1}{64}\beta\gamma\mu q^7 - \frac{1}{128}5\beta\mu q^7 - \frac{1}{32}\alpha\gamma\mu q^5 \\
&\quad + \frac{1}{16}\alpha\mu q^5 + \frac{1}{16}\gamma\mu q^3 - \frac{\mu q^3}{8} + \frac{\mu q}{2}; \\
\dot{p}_6 &= -\frac{Kp\sin\tau}{2} + \frac{Kq\cos\tau}{2} + \frac{1}{64}\beta\gamma\mu p^7 - \frac{1}{128}5\beta\mu p^7 - \frac{1}{64}\alpha\gamma\mu p^5 + \frac{1}{16}\alpha\mu p^5 - \frac{3}{32}\beta\gamma\mu p^5 q^2 - \frac{15}{128}\beta\mu p^5 q^2 \\
&\quad - \frac{\mu p^3}{8} - \frac{15}{64}\beta\gamma\mu p^3 q^4 - \frac{15}{128}\beta\mu p^3 q^4 + \frac{5}{32}\alpha\gamma\mu p^3 q^2 + \frac{1}{8}\alpha\mu p^3 q^2 + \frac{\mu p}{2} + \frac{1}{8}\beta\gamma\mu p q^6 - \frac{5}{128}\beta\mu p q^6 \\
&\quad - \frac{5}{64}\alpha\gamma\mu p q^4 + \frac{1}{16}\alpha\mu p q^4 - \frac{1}{8}\mu p q^2, \\
\dot{q}_6 &= -\frac{Kp\cos\tau}{2} - \frac{Kq\sin\tau}{2} - \frac{1}{8}\beta\gamma\mu p^6 q - \frac{5}{128}\beta\mu p^6 q + \frac{15}{64}\beta\gamma\mu p^4 q^3 - \frac{15}{128}\beta\mu p^4 q^3 + \frac{5}{64}\alpha\gamma\mu p^4 q + \frac{1}{16}\alpha\mu p^4 q \\
&\quad + \frac{3}{32}\beta\gamma\mu p^2 q^5 - \frac{15}{128}\beta\mu p^2 q^5 - \frac{5}{32}\alpha\gamma\mu p^2 q^3 + \frac{1}{8}\alpha\mu p^2 q^3 - \frac{1}{8}\mu p^2 q - \frac{1}{64}\beta\gamma\mu q^7 - \frac{1}{128}5\beta\mu q^7 + \frac{1}{64}\alpha\gamma\mu q^5 \\
&\quad + \frac{1}{16}\alpha\mu q^5 - \frac{\mu q^3}{8} + \frac{\mu q}{2}; \\
\dot{p}_8 &= -\frac{Kp\sin\tau}{2} + \frac{Kq\cos\tau}{2} + \frac{1}{256}\beta\gamma\mu p^7 - \frac{1}{128}5\beta\mu p^7 + \frac{1}{16}\alpha\mu p^5 - \frac{21}{256}\beta\gamma\mu p^5 q^2 - \frac{15}{128}\beta\mu p^5 q^2 - \frac{\mu p^3}{8} \\
&\quad + \frac{35}{256}\beta\gamma\mu p^3 q^4 - \frac{15}{128}\beta\mu p^3 q^4 + \frac{1}{8}\alpha\mu p^3 q^2 + \frac{\mu p}{2} - \frac{7}{256}\beta\gamma\mu p q^6 - \frac{5}{128}\beta\mu p q^6 + \frac{1}{16}\alpha\mu p q^4 - \frac{1}{8}\mu p q^2, \\
\dot{q}_8 &= -\frac{Kp\cos\tau}{2} - \frac{Kq\sin\tau}{2} - \frac{7}{256}\beta\gamma\mu p^6 q - \frac{5}{128}\beta\mu p^6 q + \frac{35}{256}\beta\gamma\mu p^4 q^3 - \frac{15}{128}\beta\mu p^4 q^3 + \frac{1}{16}\alpha\mu p^4 q \\
&\quad - \frac{21}{256}\beta\gamma\mu p^2 q^5 - \frac{15}{128}\beta\mu p^2 q^5 + \frac{1}{8}\alpha\mu p^2 q^3 - \frac{1}{8}\mu p^2 q + \frac{1}{256}\beta\gamma\mu q^7 - \frac{1}{128}5\beta\mu q^7 + \frac{1}{16}\alpha\mu q^5 - \frac{\mu q^3}{8} + \frac{\mu q}{2}.
\end{aligned}
$$

The subscript indicates the value of Ω in Eq. (D.3) for which the pair of above first order equations are written in (p,q) coordinates.